# A Diagram Calculus for a few Type-A Lusztig–Vogan Categories


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# Outline

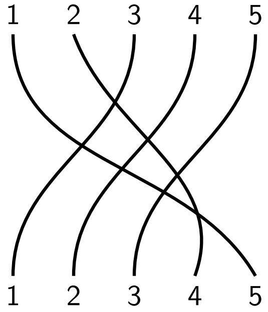
1. Why diagrammatics?
2. Diagrammatic Lusztig–Vogan categories for $\mathrm{SU}(n, 1)$
3. Links to representation theory

# Why diagrammatics?

## 1. Permutations

e.g.

$$S_5 \ni \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = (135)(24)$$



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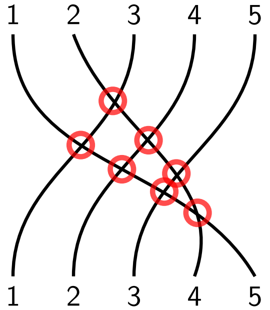
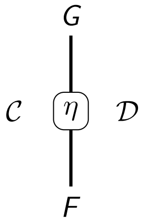



$$\operatorname{sign}\left((135)(24)\right) = (-1)^7 = -1$$

# Why diagrammatics?

**2. 2-categories**

$$\begin{array}{c} \mathcal{C} \xrightarrow{G} \mathcal{D} \\ \eta \Uparrow \\ \mathcal{C} \xrightarrow[F]{} \mathcal{D} \end{array}$$

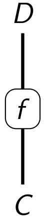



e.g. monoidal categories

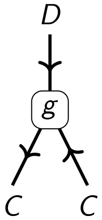
# Why diagrammatics?

**2. 2-categories: monoidal categories** (rigid, pivotal)

$$\begin{tikzcd} D \\ C \arrow[u, "f"] \end{tikzcd}$$




$$\begin{tikzcd} D \\ C \otimes C^* \arrow[u, "g"] \end{tikzcd}$$



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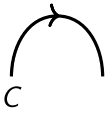
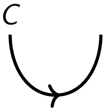
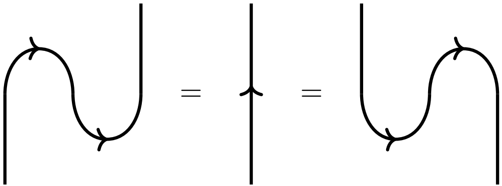



$$C^* \otimes C \to \mathbb{1}$$

,

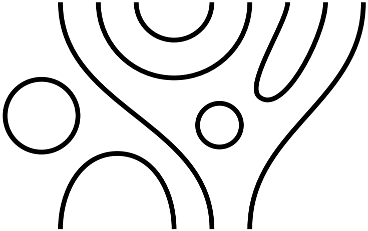



$$\mathbb{1} \to C \otimes C^*$$



# Why diagrammatics?

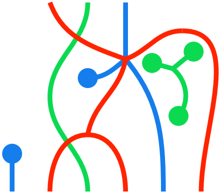
**2. 2-categories: monoidal categories** (rigid, pivotal)
e.g. the Temperley–Lieb–Jones category



Links to $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$ and produces a knot invariant.

# Why diagrammatics?

3. **Soergel bimodules** (a specific monoidal category)



- Algebraic proof of a generalisation of the Kazhdan–Lusztig conjecture [EW16]
- Counterexample to Lusztig's conjecture [Wil16]

# Diagrammatic Lusztig–Vogan category for $\mathsf{SU}(2,1)$

First off...

**Diagrammatic Soergel bimodules $\mathcal{DH}$ for**

$$S_3 = \left\langle s, t \mid s^2 = t^2 = 1, sts = tst \right\rangle$$

- $\mathbb{C}$-linear monoidal category
- *Objects:* $\mathbb{1}, {\color{red}\bullet}, {\color{blue}\bullet}, {\color{red}\bullet\bullet}, {\color{red}\bullet}{\color{blue}\bullet}, {\color{blue}\bullet}{\color{red}\bullet}, {\color{blue}\bullet\bullet}, {\color{red}\bullet\bullet\bullet}, {\color{red}\bullet\bullet}{\color{blue}\bullet}, \ldots$

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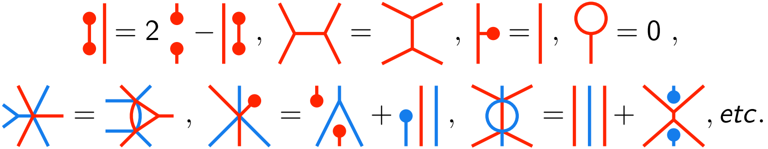
- *Relations:* Isotopy,



# Diagrammatic Lusztig–Vogan category for $\mathsf{SU}(2,1)$

$$\mathsf{SU}(2,1) = \left\{ M \in \mathsf{SL}_3(\mathbb{C}) : M^* \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

$$W = S_3 = \langle s, t \mid \ldots \rangle,$$

$$W_K = \left\langle s \mid s^2 = 1 \right\rangle \leq S_3$$

# Diagrammatic Lusztig–Vogan category for $\mathrm{SU}(2,1)$

- Right module category $\mathcal{D}\tilde{\mathcal{N}}(\mathrm{SU}(2,1)) \curvearrowleft \mathcal{D}\mathcal{H}(S_3)$

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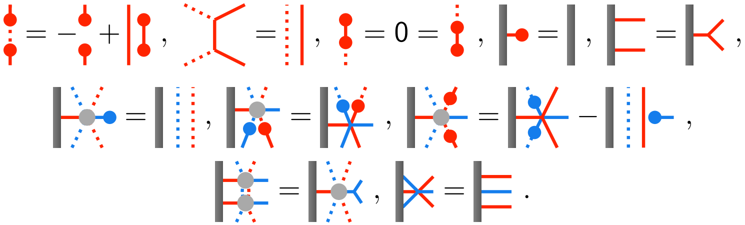
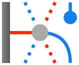
- Right module category $\mathcal{D}\tilde{\mathcal{N}}(\mathrm{SU}(2,1)) \curvearrowleft \mathcal{DH}(S_3)$
- *Objects:* generated by $\mathbb{1}, \circ, \circ\circ \qquad \curvearrowleft \mathbb{1}, \bullet, \bullet, \bullet\bullet, \dots$
- *Morphisms:* generated by

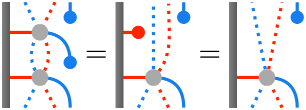
and their vertical reflections.

- *Relations:* Limited isotopy,



# Diagrammatic Lusztig–Vogan category for SU(2, 1)

e.g.  is an idempotent



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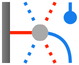
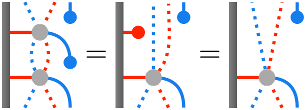
e.g.  is an idempotent



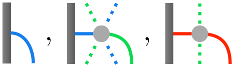
This tells us of the splitting

$$\circ\circ\bullet = \circ\circ(-1) \oplus \circ\circ(1).$$

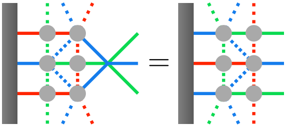
# Diagrammatic Lusztig–Vogan category for $\mathrm{SU}(2,1)$

How do we generalise this to $\mathrm{SU}(n,1)$?
e.g. $\mathcal{D}\tilde{\mathcal{N}}(\mathrm{SU}(3,1))$ with $W = S_4 = \langle s,t,u \mid \ldots\rangle$, $W_K = \langle s,t \mid \ldots\rangle$

- *Objects:* generated by $\mathbb{1}$, ○, ○○, ○○○
- *New Morphisms:*



- *New Relation:* with 3-colors



# Where is the representation theory?

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$$\begin{array}{c}\text{Monoidal structure of}\\ \text{Diagrammatic Soergel}\\ \text{bimodules } \mathcal{DH}\end{array} \overset{\text{[EW16]}}{\rightsquigarrow} \begin{array}{c}\text{Decomposition information for}\\ \text{reps of complex semisimple}\\ \text{Lie algebras}\end{array}$$

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Monoidal structure of Diagrammatic Soergel bimodules $\mathcal{DH}$ $\overset{\text{[EW16]}}{\rightsquigarrow}$ Decomposition information for reps of complex semisimple Lie algebras

Module structure of Lusztig–Vogan categories $\mathcal{N}$ $\overset{\text{[LV83]}}{\rightsquigarrow}$ Decomposition information for *admissible reps of real reductive Lie groups*

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$$\begin{array}{ccc}
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& \overset{\text{???}}{\nearrow} & \\
\text{Module structure of Diagrammatic Lusztig–Vogan categories } \mathcal{D}\tilde{\mathcal{N}} & &
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Monoidal structure of Diagrammatic Soergel bimodules $\mathcal{DH}$ $\overset{\text{[EW16]}}{\rightsquigarrow}$ Decomposition information for reps of complex semisimple Lie algebras

Module structure of Lusztig–Vogan categories $\mathcal{N}$ $\overset{\text{[LV83]}}{\rightsquigarrow}$ Decomposition information for *admissible reps of real reductive Lie groups*

Module structure of Diagrammatic Lusztig–Vogan categories $\mathcal{D}\tilde{\mathcal{N}}$ $\overset{\text{???}}{\rightsquigarrow}$

the end.