Standard filtrations & Soergel bimodules

Outline

- Standard bimodules
- Standard filtrations and soergel bimodules
- localisation

① Standard bimodules

- We stay in the usual setting:
  - fix coxeter system (W,S) ↙ reflection faithful [EMTW §5.6]
  - let $R = \mathrm{Sym}(V)$ where V is geom. rep of W i.e. $R = \mathbb{R}[\alpha_s : s \in S]$ w/ refl. action of W, $s(\alpha_t) = \alpha_t + 2\cos(\frac{\pi}{m_{st}})\alpha_t$

DEF Standard bimodules R_w as sets are just R.
As R-bimodules, has left action by multiplication in R and twisted right action

$$r \cdot x = r\, w(x)$$

where $r \in R_w$, $w \in W$, $x \in R$.

It is clear that for $v, w \in W$

$$R_v \otimes R_w \simeq R_{vw}$$

We also consider grading shifts and define

$$\mathrm{Hom}^\bullet(M,N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(M, N(i))$$

which is the set of morphisms of all degrees

PROP For $v, w \in W$, $\mathrm{Hom}^\bullet(R_v, R_w) \simeq \begin{cases} R & \text{if } x=y \\ 0 & \text{o/w} \end{cases}$

Proof
- R_v is gen by 1 as an R-bimodule (left action is enough).
  So any $\varphi \in \mathrm{Hom}(R_v, R_w)$ is determined by $\varphi(1) =: c \in R$.
- We have

$$c \cdot w(r) = c \underset{w}{\cdot} r = \varphi(1) \underset{w}{\cdot} r = \varphi(1 \underset{v}{\cdot} r) = \varphi(v(r) \cdot 1) = v(r) \cdot \varphi(1) = v(r) \cdot c$$

- R is a polynomial ring over $\mathbb{R}$ $\Rightarrow$ R is an integral domain
  so $w(r) \cdot c = v(r) \cdot c$ $\Rightarrow$ either $w(r) = v(r)$ for all $r \in R$
  or $c = 0$.
- Since W acts faithfully, the first is the same as $w = v$.
  In which case we have a bijection

$$\varphi \in \mathrm{Hom}^\bullet(R_v, R_w) \longleftrightarrow c \in R. \qquad \square$$

Let $\mathrm{StdBim} \hookrightarrow R\text{-gbim}$ be the full subcategory gen. by $\{R_w : w \in W\}$ closed under $\oplus$ and grading shifts. This is automatically
- monoidal
- closed under direct summands
  - bc. we only need to look at R_w and it has idempotents in

$$\mathrm{Hom}^\bullet(R_w, R_w) \simeq R = \mathbb{R}[\alpha_s : s \in S]$$

  which are idempotents of $\mathbb{R}$, which is only the identity map.
  - (though not semisimple eg. schur's lemma does not hold)

The split Grothendieck group is $[\mathrm{StdBim}]_\oplus = \mathbb{Z}[v^{\pm 1}][W]$ the group algebra.

There is essentially nothing interesting here

Compare $\mathrm{SBim} \rightsquigarrow \mathcal{H}$ Hecke algebra

$\uparrow$ deformation

$\mathrm{StdBim} \rightsquigarrow \mathbb{Z}[v^{\pm 1}][W]$ group algebra

② Standard filtrations and Soergel bimodules

Recap on Soergel bimodules

- let $B_s := R \otimes_{R^s} R(1)$ for $s \in S$

  $BS(\underline{w}) := B_{s_1} \otimes \cdots \otimes B_{s_n}$ for $\underline{w} = (s_1, \dots, s_n)$

  $\simeq R \otimes_{R^{s_1}} \cdots \otimes_{R^{s_n}} R(n)$

- A Soergel bimodule is a direct summand of some $BS(\underline{w})$

let $\overset{\text{degree } -1}{c_{id} = 1 \otimes_s 1}$, $\overset{\text{degree } 1}{c_s = \frac{1}{2}(\alpha_s \otimes_s 1 + 1 \otimes_s \alpha_s)}$ be elements in B_s.

FACT (EMTW, Exercise 4.28) For any $f \in R$,

$f \cdot c_{id} = c_{id} \cdot s(f) + c_s \cdot \partial_s(f)$ and $f \cdot c_s = c_s \cdot f$

Also $\{c_{id}, c_s\}$ is a basis for B_s as a left or right R-module.

PROOF

- If $f \in R^s$, then $f \cdot c_{id} = c_{id} \cdot f + c_s \cdot \underbrace{\partial_s(f)}_{0}$

  and $f \cdot c_s = c_s \cdot f$ trivially.

- If $f = \alpha_s$, then

$$c_{id} \cdot f + c_s \cdot \partial_s(f) = -1 \otimes_s \alpha_s + \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) \cdot 2$$
$$= \alpha_s \otimes 1$$
$$= f \cdot c_{id}$$

and $f c_s - c_s f = \frac{1}{2}(\cancel{\alpha_s^2 \otimes_s 1} + \cancel{\alpha_s \otimes_s \alpha_s} - \cancel{\alpha_s \otimes_s \alpha_s} - \cancel{1 \otimes_s \alpha_s^2})$
$= 0.$

$g = \partial_s(f \frac{\alpha_s}{2})$
$h = \frac{1}{2}\partial_s(f)$

- For general $f \in R$, we can write $f = g + h\alpha_s$ for $g, h \in R^s$.

  Then $c_{id} \cdot f + c_s \cdot \partial_s(f) = c_{id} \cdot s(g + h\alpha_s) + c_s \cdot g + c_s \cdot h\, \partial_s(\alpha_s)$

  $= g \cdot c_{id} + h(c_{id} \cdot s(\alpha_s) + c_s \cdot \partial_s(\alpha_s))$

  $= (g + h\alpha_s) \cdot c_{id}$

  and $f \cdot c_s = (g + h\alpha_s) \cdot c_s = c_s \cdot (g + h\alpha_s) = c_s \cdot f$.

- Now, the first equality shows $\{c_{id}, c_s\}$ is basis of B_s as right R-module.

  Putting $s(f)$ & rearranging we get

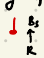
  $c_{id} \cdot f = s(f) \cdot c_{id} - c_s \cdot \partial_s(s(f))$

  $= s(f) \cdot c_{id} + \partial_s(f) \cdot c_s$

  so we have $\{c_{id}, c_s\}$ is basis for B_s as left R-module.

□

By "$f c_s = c_s f$", c_s generates a copy of $R \simeq \{f c_s : f \in R\}$ inside B_s.

ie.
$$R(-1) \xhookrightarrow{1 \mapsto c_s} B_s$$
(diagram is )

coming from the Frobenius alg. structure.

The cokernel is isomorphic to $R_s(1)$ with the map

$$\mu_s : B_s \twoheadrightarrow R_s(1)$$
$$f \otimes g \longmapsto f s(g)$$

$\mu_s : c_{id} \longmapsto 1$

(see this by $B_s/\langle c_s \rangle$ has $f \cdot c_{id} = c_{id} \cdot s(f)$ generated by c_{id})

So we get a short exact sequence

$$0 \longrightarrow R(-1) \xrightarrow{1 \mapsto c_s} B_s \xrightarrow{c_{id} \mapsto 1} R_s(1) \longrightarrow 0 \qquad (\nabla)$$

FACT (EMTW, Exercise 5.6)

Let $d_s = \frac{1}{2}(\alpha_s \otimes 1 - 1 \otimes \alpha_s)$. Then

$$f \cdot c_{id} = c_{id} \cdot f + d_s \cdot \partial_s(f), \qquad f \cdot d_s = d_s \cdot s(f)$$

and $\{c_{id}, d_s\}$ is a basis for B_s as a left (or right) R-module.

Proved similarly to $\{c_{id}, c_s\}$.

By $f \cdot d_s = d_s \cdot s(f)$, d_s generates a copy of $R_s(-1)$ inside B_s

$$R_s \xhookrightarrow{1 \mapsto d_s} B_s \qquad \left(\text{diagram in } \begin{matrix} B_s \\ \uparrow \\ R_s \end{matrix}\right)$$

this has a cokernel iso. to $R(1)$ by the map

$$\begin{aligned} \mu_{id} : B_s &\twoheadrightarrow R(1) \\ f \otimes g &\mapsto fg \\ c_{id} &\mapsto 1 \end{aligned}$$

(because in $B_s/\langle d_s \rangle$, $f \cdot c_{id} = c_{id} \cdot f$.)

So we get a short exact sequence

$$0 \to R_s(-1) \xrightarrow{1 \mapsto d_s} B_s \xrightarrow{c_{id} \mapsto 1} R(1) \to 0 \qquad (\Delta)$$

<u>Notice</u> B_s is filtered by R and R_s in no particular order, and grading shift appearing depends on order.

We can extend to say any $BS(\underline{w})$ has a filtration by standards

eg. $\underline{w} = (s, s)$

$(\Delta \otimes B_s)$ $\quad 0 \to R_s B_s(-1) \to B_s B_s \xrightarrow{\psi} R B_s(1) \to 0$

$(R_s(-1) \otimes \Delta)$ $\quad 0 \to \underset{R(-2)}{\underset{\|}{R_s R_s(-2)}} \to R_s B_s(-1) \to \underset{R_s}{\underset{\|}{R_s R}} \to 0$

$(R(1) \otimes \Delta)$ $\quad 0 \to \underset{R_s}{\underset{\|}{R R_s}} \to R B_s(1) \to \underset{R(2)}{\underset{\|}{R R(2)}} \to 0$

<u>Note</u> If we tensor by free left or right R-module then it is exact (since free $\Rightarrow$ projective)

Also these $BS(\underline{w})$ are all free as left or right R-modules

Gives filtration

$$0 \subset \underset{R_s}{R(-2)} \subset \underset{RR_s \simeq R_s}{R_s B_s(-1)} \subset \underset{R(2)}{\psi^{-1}(RR_s)} \subset B_s B_s$$

$$\left(\begin{tikzcd} 0 \subset R(2) \subset R_s B_s(-1) \arrow[r, hook] \arrow[dr, "\sim"'] & B_s B_s \arrow[r, two heads, "\psi"] & R B_s(1) \\ & \psi^{-1}(RR_s) \arrow[r, two heads] \arrow[u, phantom, "\cup"] & R R_s \arrow[u, phantom, "\cup\ R(2)"] \\ & \psi^{-1}(0) \arrow[r, two heads] \arrow[u, phantom, "\cup"] & 0 \arrow[u, phantom, "\cup\ RR_s"] \end{tikzcd} \rightsquigarrow \begin{aligned} 1 \otimes_s 1 \otimes_s f &\mapsto 1 \otimes_s f \\ \psi^{-1}(RR_s) &= \left\langle \begin{matrix} \alpha_s \otimes 1 \otimes 1 - 1 \otimes 1 \otimes \alpha_s, \\ 1 \otimes \alpha_s \otimes 1 - 1 \otimes 1 \otimes \alpha_s \end{matrix} \right\rangle \\ &= \langle 1 \otimes d_s, c_s \otimes 1 \rangle \end{aligned}\right)$$

<u>PROP</u> In an abelian category let

$$\begin{matrix} \psi : A & \twoheadrightarrow & B \\ \cup & & \cup \\ A' & & B' \end{matrix}$$

Then $\psi^{-1}(B)/\psi^{-1}(B') \simeq B/B'$.

<u>PROOF</u> We have this diagram

$$\begin{tikzcd} B' \arrow[r, hook] & B \arrow[r, two heads] & B/B' \\ \psi^{-1}(B') \arrow[r, hook] \arrow[u, "\psi"] & A \arrow[r, two heads] \arrow[u, "\psi"] \arrow[ur] & \psi^{-1}(B)/\psi^{-1}(B') \arrow[u, dotted, "\psi"'] \end{tikzcd}$$

pullbacks preserve mono.

Where the left square is a pullback square. By stacks lemma 12.5.13 (08N4) this is also pushout square. By stacks lemma 12.5.12 (08N3), ψ is an iso. $\square$

Hence any direct summand of $BS(\underline{w})$ has such a fitration.

- The subquotients look like tensors of some $R(1)$ and some $R_{s_i}(-1)$
- <u>note</u> this corresponds to subexpressions $\underline{e} \subset \underline{w}$ where choosing $R(1) \leftrightarrow e_i = 0$, $R_{s_i}(-1) \leftrightarrow e_i = 1$

Since R_w is indecomposable, every Soergel Bimodule has a fitration by standards

However, there are many ways to build this fitration, also can use Δ, ∇ or a mix, also gradings appearing may change depending on how we build it, also the order of subquotients may not respect the Bruhat order.

Fix an enumeration of W, $x_0, x_1, \dots$ such that $x_i \le x_j$ in Bruhat order $\Rightarrow i \le j$.

eg for A_2, $1 < s < t < st < ts < sts$

which refines the Bruhat order.

The problems above can be fixed by the deep result of Soergel

THM (Soergel) For such a fixed enumeration of W,

any Soergel bim B has a unique filtration $0 = B^k \subset B^{k-1} \subset \dots \subset B^0 = B$ s.t.

$$B^i / B^{i+1} \simeq R_{x_i}^{\oplus h_{x_i}}$$ ← subquotients appear in decreasing order

where $h_{x_i} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. (called a Δ-filtration)

Further, for any $x \in W$, the graded multiplicity h_x depends only on B and x (not on choice of enum of W)

NOTE "having a Δ-fitration is" closed under $\otimes$ and $\oplus$

eg B_s has Δ-fitration

$$0 \subset \underset{= R_s^{\oplus v^{-1}}}{\underset{R_s(-1)}{R_s(-1)}} \subset \underset{= R^{\oplus v}}{\underset{R(1)}{B}}$$

DEF The Δ-character of Soergel bimodule B is

$$ch_\Delta(B) := \sum_{x \in W} v^{\ell(x)} h_x(B)\, \delta_x \quad \in \mathcal{H}.$$

eg $ch_\Delta(B_s) = v^{\ell(s)} v^{-1} \delta_s + v^{\ell(1)} v\, \delta_{id}$

$= v + \delta_s$

We can define also ∇-fitrations (call gr. mult. h'_{x_i}) w/ subquotients increasing in order and the same result holds.

DEF The ∇-character of Soergel bimodule B is

$$ch_\nabla(B) := \sum_{x \in W} v^{\ell(x)} \overline{h'_x(B)}\, \delta_x \quad \in \mathcal{H}$$

FACTS

- $ch_\Delta(B \oplus B') = ch_\Delta B + ch_\Delta B'$, $\quad ch_\nabla(B \oplus B') = ch_\nabla B + ch_\nabla B'$
- $ch_\Delta(B(1)) = v\, ch_\Delta B$, $\quad ch_\nabla(B(1)) = v^{-1} ch_\nabla B$

$\rightsquigarrow ch_\Delta, ch_\nabla : [\mathrm{SBim}]_\oplus \longrightarrow \mathcal{H}$

but only ch_Δ is $\mathbb{Z}[v^{\pm 1}]$-linear, so define $ch := ch_\Delta$.

THM (Soergel categorification thm)

① $\{\text{Indecomp of SBim}\}/_{\simeq,(1)} \xleftrightarrow{1-1} W$

$\qquad B_w \longleftrightarrow w$

② $ch : [\mathrm{SBim}]_\oplus \longrightarrow \mathcal{H}$ is an isomorphism

THM (Soergel's conjecture) For $x \in W$,

$ch(B_x) = b_x$ and hence KL poly $h_{x,y} = h_x(B_y)$

③ Localisation

Let $Q := \mathrm{Frac}(R)$ fraction field. We don't consider this as graded.

Let $BS(\underline{w})_Q := BS(\underline{w}) \otimes_R Q$.

<u>PROP</u> $BS(\underline{w})_Q = (R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_n}} R(n)) \otimes_R Q = Q \otimes_{Q^{s_1}} Q \otimes_{Q^{s_2}} \cdots \otimes_{Q^{s_n}} Q$

<u>Proof</u>

First we show $\overset{B_{s,Q} :=}{B_s \otimes Q} = R \otimes_{R^s} Q \simeq Q \otimes_{Q^s} Q$.

- Clearly $R \otimes_{R^s} Q \hookrightarrow Q \otimes_{Q^s} Q$, so we show this is surjective.
- Enough to show we can get $\frac{1}{f} \otimes_s 1$ for any $f \neq 0$.

But $\frac{1}{f} \otimes 1 = \frac{s(f)}{f s(f)} \otimes 1 = s(f) \otimes \frac{1}{f s(f)} \in R \otimes_{R^s} Q$

By induction the result holds. ∎

- Similarly $R_s \otimes Q \simeq Q_s$ for standard bimodules.

<u>Note</u>

- We see changing scalars from R to Q lands in Q-bim
- If M is free R-module then $M \hookrightarrow M \otimes_R Q$ since M is free (each R includes into Q)
  We know $\mathrm{Hom}(B,B')$ is free R-module for $B, B' \in \mathrm{SBim}$, so $\mathrm{Hom}(B,B') \hookrightarrow \mathrm{Hom}(B,B') \otimes_R Q$
  ie. localisation is faithful functor!
- Q is flat (a field) so $- \otimes Q$ is exact on R-modules
  so we has ses

$$(\Delta \otimes Q) \quad \begin{tikzcd} 0 \arrow[r] & Q_s \arrow[r, "1 \mapsto d_s"] & B_{s,Q} \arrow[r, "c_{id} \mapsto 1"] & Q \arrow[r] \arrow[l, bend left, "\frac{1}{\alpha_s} c_s \mapsfrom 1"] & 0 \end{tikzcd}$$

$$\text{or } (\nabla \otimes Q) \quad \begin{tikzcd} 0 \arrow[r] & Q \arrow[r, "1 \mapsto c_s"] & B_{s,Q} \arrow[r, "c_{id} \mapsto 1"] & Q_s \arrow[r] \arrow[l, bend left, "\frac{1}{\alpha_s} d_s \mapsfrom 1"] & 0 \end{tikzcd}$$

which split. So $B_{s,Q} = Q \oplus Q_s$.

Therefore every Soergel bimodule over Q splits into standards

Particularly

$$BS(\underline{w})_Q = \bigoplus_{\underline{e} \subset \underline{w}} Q_{\underline{w}^{\underline{e}}}$$

We have $\mathrm{Hom}(Q_v, Q_w) = \begin{cases} Q & \text{if } v = w \\ 0 & \text{o/w} \end{cases}$

so it is easy to check isomorphism. We can check non-isomorphism of objects from SBim by passing through the localisation functor.

- On morphism, faithfulness means we can check equality of morphisms in SBim by checking in localisation

Category stuff

- Let $\mathrm{BSBim}_Q \longrightarrow Q\text{-bim}$ full subcat gen. by $BS(\underline{w})_Q$ for $w \in W$.
  $\mathrm{SBim}_Q := \mathrm{Kar}(\mathrm{BSBim}_Q)$
  $\mathrm{StdBim}_Q \longrightarrow Q\text{-bim}$ full subcat gen by Q_w for $w \in W$
- Then

$$\begin{tikzcd} \mathrm{BSBim} \arrow[r, "\mathrm{Kar}"] \arrow[d, "-\otimes Q"', "\text{faithful}"] & \mathrm{SBim} \arrow[d, "\mathrm{loc}"', "\text{faithful}"] & \mathrm{StdBim} \arrow[d, "-\otimes Q", "\text{faithful}"'] \\ \mathrm{BSBim}_Q \arrow[r, "\mathrm{Kar}"'] & \mathrm{SBim}_Q \arrow[r, "\sim"'] & \mathrm{StdBim}_Q \end{tikzcd}$$

- Since we lose gradings in StdBim_Q, $[\mathrm{StdBim}_Q]_\oplus \simeq \mathbb{Z}[W]$. So we have

$$\begin{tikzcd} \mathrm{SBim} \arrow[r, squiggly, "\mathrm{ch}"] \arrow[d, "\mathrm{loc}"'] & \mathcal{H}(W,S) \arrow[d, "v \mapsto 1"] \\ \mathrm{StdBim}_Q \arrow[r, squiggly, "\mathrm{ch}"'] & \mathbb{Z}[W] \end{tikzcd}$$

"localisation in categorification of specialisation at v=1"