OTI Talk notes

GOAL Structure of rep of $\Bbbk S_n$ in char p (over all n)

- Basic representation of $\widehat{sl}_p$ (type $A^{(1)}_{p-1}$ Kac-Moody)
- Branching behaviour & crystals
- Blocks & weight spaces
- Timing 45–60 min

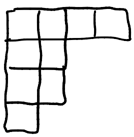
Outline

- Recap on Rep S_n in characteristic 0
- Characteristic p (p prime)
  - Initial problems
  - Block decomposition ← formal characters?
  - Induction and restriction
    + problems in characteristic p (see Hecke Crystals paper)
  - introduce connection to $\widehat{sl}_p$
- To $\widehat{sl}_p$ (Kac-Moody $A^{(1)}_{p-1}$)
  - Definition
  - Cartan matrix
  - Basic representation
  - Connection to Rep S_n in char p (details explained below)
  - Crystal of the basic representation + modular branching graph
    ← tangent to meaning
  - Weight spaces of $V(\Lambda_0)$ + blocks

1. Characteristic 0 representations of S_n

- Let char $\Bbbk = 0$; let $n \geq 1$
- A partition of n is an integer sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ s.t $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$ and $\sum \lambda_i = n$. We may visualise partitions with Young diagrams.

eg. $n = 9$, $\lambda = (4,2,2,1)$



- A (Young) tableau is an injective filling of these boxes with $\{1,2,\dots,n\}$

eg

3	5	4	8
9	1		
7	2		
6			

A standard (Young) tableau is a tableau where rows increase from left to right, and columns increase from top to bottom.

eg

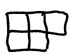
1	2	4	8
3	6		
5	9		
7			

standard

1	2	4	8
3	9		
5	6		
7			

not standard

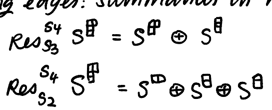
- Irreducible representations of S_n are indexed by partitions λ of n and has basis of standard tableaux for λ. We write S^λ for the corresponding irrep. (called a Specht module).

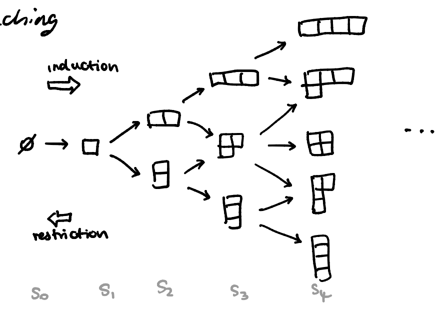
eg. $n=5$, $\lambda =$ 

$$S^\lambda = \mathrm{span}_{\Bbbk}\left\{ \begin{smallmatrix}1&2&3\\4&5\end{smallmatrix},\ \begin{smallmatrix}1&2&4\\3&5\end{smallmatrix},\ \begin{smallmatrix}1&2&5\\3&4\end{smallmatrix},\ \begin{smallmatrix}1&3&4\\2&5\end{smallmatrix},\ \begin{smallmatrix}1&3&5\\2&4\end{smallmatrix} \right\}$$

The action of S_n is not simple (essentially "permuting the numbers" with some rules)
- details eg. via considering polytabloids

- Branching information is summed up in the branching diagram.
  - nodes are irreps of S_n (the columns) given by partitions of n
  - left is restriction, right is induction
  - Outgoing edges: summands in induction ($S_n \rightsquigarrow S_{n+1}$)
    Incoming edges: summands in restriction ($S_n \rightsquigarrow S_{n-1}$)

    eg. 



2. Characteristic $p>0$ representations of S_n

- Let char $\Bbbk = p > 0$ and $n \geq 1$.

- Initial problems
  - S^λ may no longer be irreducible
  - Maschke's theorem no longer holds, so rep. theory is not semisimple

- We take a more Lie theoretical approach

DEF For $k = 1, \dots, n$ the Jucys-Murphy elements are

$$x_k := \sum_{i=1}^{k-1} (i\ k) \qquad \in \Bbbk S_n$$

eg. $x_1 = 0$
$x_2 = (1\ 2)$
$x_3 = (1\ 3) + (2\ 3)$
$x_4 = (1\ 4) + (2\ 4) + (3\ 4)$

It is easy to see they commute with each other:
- WLOG suppose $i>j$. Notice x_i is fixed by permutation of elements $<i$. In particular it is fixed by conjugation not involving i. Then all terms of x_j commute with x_i, so $x_i x_j = x_j x_i$
- It is obvious when $i=j$.

Let M be an $\Bbbk S_n$ module, $I = \mathbb{Z}/p\mathbb{Z} \subseteq \Bbbk$ and $\underset{\sim}{i} = (i_1, \dots, i_n) \in I^n$.

DEF The simultaneous generalised eigenspace of M (corresp. to $x_1, \dots, x_n$ w/ eigenvalues $i_1, \dots, i_n$) is $M[\underset{\sim}{i}] := \{ v \in M : (x_k - i_k)^N v = 0 \text{ for some } N>0,\ k = 1, \dots, n\}$

LEM Any $\Bbbk S_n$ module M can be decomposed $M = \bigoplus_{\underset{\sim}{i} \in I^n} M[\underset{\sim}{i}]$ as vector spaces (not necessarily $\Bbbk S_n$-modules)
- The proof just says I^n contains all possible eigenvalues.

- Given $\underset{\sim}{i} \in I^n$, it's weight is $\mathrm{wt}(\underset{\sim}{i}) = \gamma = (\gamma_i)_{i \in I}$ where γ_i is number of i's in $\underset{\sim}{i}$.

  eg. $p=5$, $n=4$, $\underset{\sim}{i} = (1,2,3,3)$ then $\mathrm{wt}(\underset{\sim}{i}) = (\underset{0}{0}, \underset{1}{1}, \underset{2}{1}, \underset{3}{2}, \underset{4}{0})$
  - This groups eigenvalues $\underset{\sim}{i} \in I^n$ into S_n-orbits (by permuting entries)
  - Write Γ_n be the set of possible γ

Let $\gamma \in \Gamma_n$, then write $M[\gamma] := \bigoplus_{\substack{\underset{\sim}{i} \in I^n \\ \mathrm{wt}(\underset{\sim}{i}) = \gamma}} M[\underset{\sim}{i}]$. This is a $\Bbbk S_n$ module unlike $M[\underset{\sim}{i}]$ by itself.

We call γ blocks and $M = \bigoplus_{\gamma \in \Gamma_n} M[\gamma]$ the block decomposition of M into $\Bbbk S_n$ modules
When $M = M[\gamma]$ we say M belongs to block γ.

- Induction and restriction
  - For $\gamma \in \Gamma_n$ and $i \in I$, let $\gamma + i$ be $(\gamma_1, \dots, \gamma_i + 1, \dots)$ ← only i^{th} one is different

    and if $\gamma_i > 1$, let $\gamma - i$ be $(\gamma_1, \dots, \gamma_i - 1, \dots)$ ← only i^{th} one is different
  - For M in block γ define:

    i-induction $\quad f_i M := (\mathrm{Ind}_{S_{n-1}}^{S_n} M)[\gamma + i]$

    and i-restriction $\quad e_i M := \begin{cases} (\mathrm{Res}_{S_{n-1}}^{S_n} M)[\gamma - i] & \text{if } \gamma_i > 1 \\ 0 & \text{o/w} \end{cases}$
  - Extend additively to any $\mathbb{k}S_n$-module M. Infact these induce exact functors $e_i : \mathbb{k}S_n\text{-mod} \longrightarrow \mathbb{k}S_{n-1}\text{-mod}$, $f_i : \mathbb{k}S_n\text{-mod} \longrightarrow \mathbb{k}S_{n+1}\text{-mod}$

<u>EDIT</u> The way the JM elems x_n act on the induced module

$$\mathrm{Ind}_{S_{n-1}}^{S_n} M = \mathbb{k}S_n \otimes_{\mathbb{k}S_{n-1}} M$$

is by the endomorphism

$$g \otimes m \longmapsto g x_n \otimes m.$$

This clearly commutes with the left $\mathbb{k}S_n$ action and is well defined since if $h \in \mathbb{k}S_{n-1}$,

$$\begin{array}{ccc} h \otimes m & = & 1 \otimes hm \\ {\scriptstyle x_n} \downarrow & & \downarrow {\scriptstyle x_n} \\ h x_n \otimes m & = & x_n \otimes hm \end{array}$$

← since $x_n \in C_{\mathbb{k}S_n}(\mathbb{k}S_{n-1})$.

  - f_i induces and keeps the blocks where an i-eigenvalue was added
  - e_i restricts and keeps eigenspaces where x_n acted with eigenvalue i

e.g. $n = 4$, $p = 3$

$$M = \bigoplus_{\underline{i} \in I^n} M[\underline{i}] = M[(1,2,2,1)] \oplus M[(2,1,1,2)] \oplus M[(1,2,1,2)]$$

$$e_0 M = 0$$
$$e_1 M = M[(1,2,2)]$$
$$e_2 M = M[(2,1,1)] \oplus M[(1,2,1)]$$

- Grothendieck groups $\mathrm{Gr}(\mathbb{k}S_n\text{-mod})$

  Let $\mathcal{G} = \mathbb{C} \otimes_{\mathbb{Z}} \left(\bigoplus_{n \geq 0} \mathrm{Gr}(\mathbb{k}S_n\text{-mod}) \right)$ and extend e_i, f_i to $\mathbb{C}$-linear operators

<u>THM</u> (Lascoux-Leclerc-Thibon, 1996)

- The operators e_i and f_i on $\mathcal{G}$ (for $i \in I$) satisfy the relations of Chevalley generators of the affine Lie algebra $\widehat{\mathfrak{sl}_p}$.
- As an $\widehat{\mathfrak{sl}_p}$-module, $\mathcal{G}$ is isomorphic to the basic representation $V(\Lambda_0)$ of $\widehat{\mathfrak{sl}_p}$ generated by the highest weight vector corresponding to trivial rep of S_0.
- The decomposition of $\mathcal{G}$ into blocks coincides with the weight space decomp of $V(\Lambda_0)$.

3. To the affine Lie algebra $\widehat{sl_p}$

DEF The affine Lie algebra $\widehat{sl_p}$ is the Lie algebra generated by E_i, F_i, H_i for $i = 0, \dots, p-1$ under the relations

- $[H_i, H_j] = 0 \quad \forall i,j$
- $[H_i, E_j] = a_{ij} E_j \quad \forall i,j$
- $[H_i, F_j] = -a_{ij} F_j \quad \forall i,j$
- $[E_i, F_j] = \delta_{ij} H_i \quad \forall i,j$, δ_{ij} Kronecker Delta
- $(\mathrm{ad}\, E_i)^{1-a_{ij}}(E_j) = 0 \quad \forall i \neq j$
- $(\mathrm{ad}\, F_i)^{1-a_{ij}}(F_j) = 0 \quad \forall i \neq j$

Where (a_{ij}) is the generalised Cartan matrix $\underbrace{\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}}_{p \times p}$ if $p \neq 2$ and $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ when $p = 2$.

- There is an extra root that interacts with the first and last root of sl_p
  - see Cartan matrix or affine Dynkin diagram $\tilde{A}_{p-1}$

- Given a weight $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{p-1}) \in \mathbb{Z}^{p-1}$ the highest weight representation $V(\gamma)$ is a representation such that there exists $v_0 \in V$ ("highest weight vector") s.t.
$$E_i v_0 = 0 \quad \forall i$$
$$H_i v_0 = \gamma_i v_0 \quad \forall i$$
and $\mathcal{U}(\widehat{sl_p})\, v_0 = V$

The basic representation is the module with heighest weight $\Lambda_0 = (1, 0, \dots, 0)$
- In other words $V(\Lambda_0)$ is generated by a highest weight vector v_0 such that
$$E_i v_0 = 0 \quad \text{and} \quad H_i v_0 = \begin{cases} v_0 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$
- The vector space is generated by F_i action on v_0 subject to the last relations of $\widehat{sl_p}$
- Indeed by the theorem we saw, this can be realised as $\mathcal{G}$ where E_i acts by e_i and F_i acts by f_i and v_0 is the trivial FS_0-module

4. Crystals

- Associated to the basic representation is a combinatorial object called a crystal (defined by Kashiwara 1990)
  - we describe the crystal and the connection to FS_n-modules

- The following explicit description of the crystal for the basis rep of $\widehat{sl_p}$ (Misra-Miwa 1990)
  - For any partitions we can define

DEF The residue of a box $B = (a, b)$ in a partition diagram is $\mathrm{res}\, B = (b - a) \bmod p$.

We will see this is analogous to content vectors/spectra of JM elem as in char 0 reps.

eg. $p = 3$, $\lambda = (4, 3, 1)$

(a) \ (b)	0	1	2	3
0	0	1	2	0
1	2	0	1	
2	1			

DEF Given residue $i \in I = \mathbb{Z}/p\mathbb{Z}$ and partition, a box B
- i-removable if $B \in \lambda$, $\mathrm{res}\, B = i$ and $\lambda \setminus B$ is a partition
- i-addable if $B \notin \lambda$, $\mathrm{res}\, B = i$ and $\lambda \cup B$ is a partition

eg.

0	1	2	0
2	0	1	
1			

1-addable

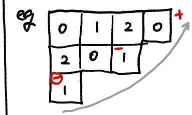
1-removable

DEF Identifying all the i-addable and i-removable boxes of λ

- the <u>i-signature</u> of λ is a sequence of $\{+,-\}$ corresp to addable & removable boxes, from bottom left to top right
- the <u>reduced i-signature</u> of λ is obtained by "cancelling" $-+$ recursively in the i-signature.
  - The $-$'s are called <u>i-normal</u> and $+$'s are called <u>i-conormal</u>
  - The leftmost $-$ is called <u>i-good</u> and rightmost $+$ is called <u>i-cogood</u>

Of course reduced signatures are always $+$'s followed by $-$'s (if both present)

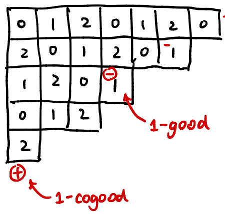
eg



i-signature $= - - +$

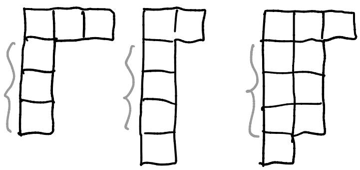
red. i-sign. $= - \cancel{-}\cancel{+} = -$

<u>Example</u>

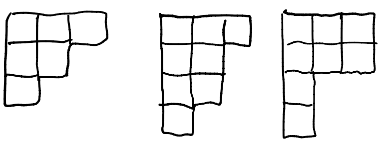


DEF A <u>p-singular partition</u> is one containing p non-zero equal parts.
A <u>p-regular partition</u> is a non-p-singular partition.

<u>Example</u> 3-singular partitions

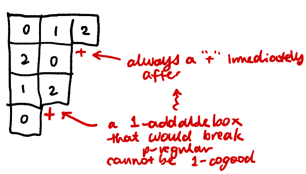


3-regular partitions



<u>Note</u> adding i-cogood and removing i-good boxes from a p-regular partition gives a p-regular partition

eg. $p=3$

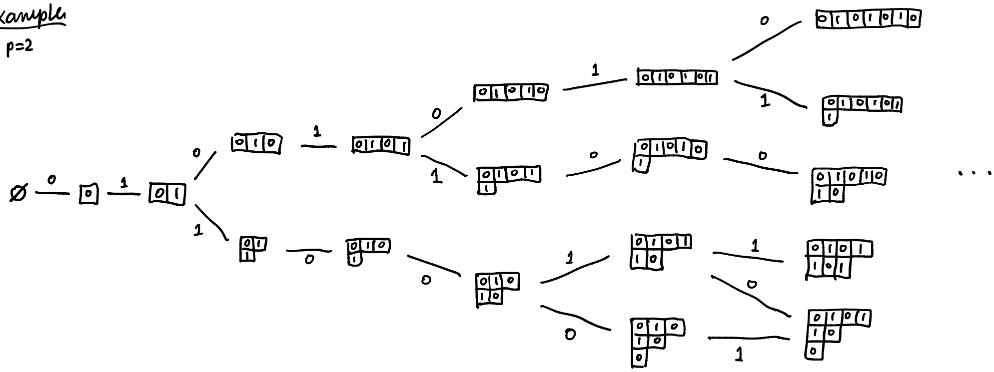


- Crystal graph
  - nodes = p-regular partitions
  - labelled edge $\lambda \xrightarrow{i} \mu$ = i-cogood box added to λ

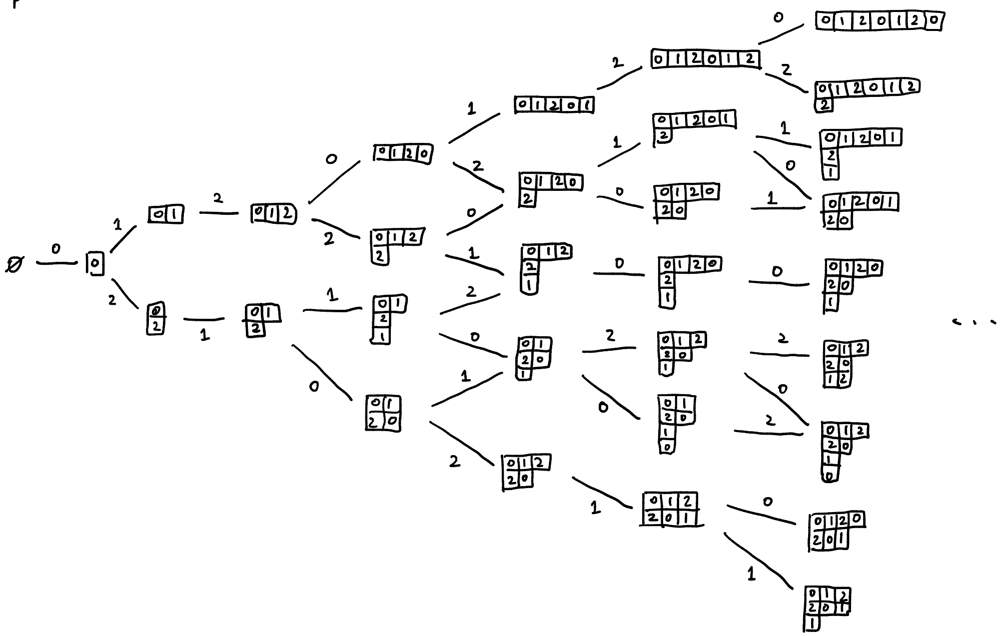
  (Since there is a unique i-cogood box, we get at most 1 outgoing i-edge and ——— ingoing i-edge)

Example

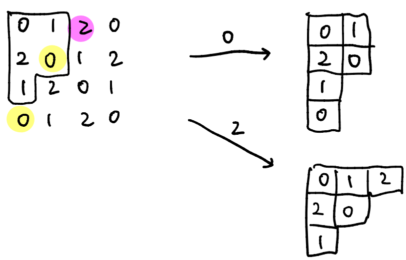
- $p=2$



- $p = 3$



eg



- Modular branching graph
  - the p-residues remind us of the content vectors from char 0 reps
    ie. corresponding to eigenvalues of JM-elements

LEM (Nakayama 1996)
If D is an irred. $\Bbbk S_n$-module and $i \in I = \mathbb{Z}/p\mathbb{Z}$, then $e_i D$ (resp $f_i D$) is either zero or self-dual $\Bbbk S_{n-1}$ (resp $\Bbbk S_{n+1}$) -module with irreducible socle $\simeq$ head.

largest semisimple submodule

largest semisimple quotient

Write $\tilde{f}_i = \mathrm{soc} \circ f_i$ and $\tilde{e}_i = \mathrm{soc} \circ e_i$

DEF The modular branching graph has
- vertices = iso classes of irred $\Bbbk S_n$-module for all $n \geq 0$
- edge $D \xrightarrow{i} E$ if $E = \tilde{f}_i D$ (or equivalently $D = \tilde{e}_i E$)

Note this is not the branching graph, but gives us a peek at it

THM (Lascoux–Leclerc–Thibon 1996)
The modular branching graph is uniquely isomorphic (as $\mathbb{Z}/p\mathbb{Z}$-labelled digraph) to the crystal graph of basic rep of $\widehat{\mathfrak{sl}}_p$.

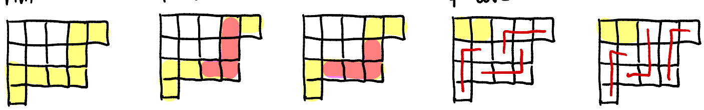
- This implies irreducible $\Bbbk S_n$-modules are parametrised by p-regular partitions
  ie. choosing a path $\varnothing \xrightarrow{i_1} \square \xrightarrow{i_2} \cdots \xrightarrow{i_n} \lambda$ to p-regular partition λ,
  corresp irred $D^\lambda := \tilde{f}_{i_n} \cdots \tilde{f}_{i_1} D^\varnothing$ where $D^\varnothing$ is trivial $\Bbbk S_0$-module.

5. Weight spaces

- The decomposition of the basic representation $V(\Lambda_0)$ into weight spaces can be described combinatorially with partitions.

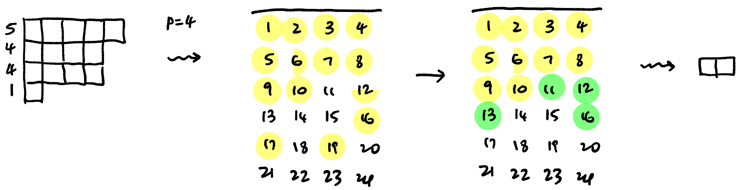
DEF For any partition λ
- the rim of λ are the boxes s.t. there is no box in at least one of S, E, SE directions.
- a p-hook is a conected part of the rim, consisting of p boxes.
- the p-core is the partition remaining after iteratively removing p-hooks s.t. the remainder is a partition
- the p-weight is the number of p-hooks removed to obtain p-core

eg rim 4-hook 4-core 4-weight = 3



(can be done with any p, but we are interested when $p = \mathrm{char}\, \Bbbk$)

It can be shown that p-core & p-weight are unique for a given partition (See James' Abacus)



The pairs (p-core, p-weight) parametrise the weight spaces of $V(\Lambda_0)$

- Blocks of $\Bbbk S_n$-modules
  - Recall that blocks are indexed by tuples γ that count the number of generalised eigenvalues of each $i \in I = \mathbb{Z}/p\mathbb{Z}$ appearing in generalised eigenspace decomps $M = \bigoplus_{\underline{i} \in I^n} M[\underline{i}]$
  - The theorem from the start says there is a 1-1 correspondence between blocks and (non-zero) weightspaces of $V(\Lambda_0)$
  - A consequence of this is another proof of the Nakayama conjecture

    | For $n \geq 0$, the irreducible $\Bbbk S_n$-modules D^λ and D^μ (λ, μ p-regular) belong to the same block iff λ, μ have the same p-core.

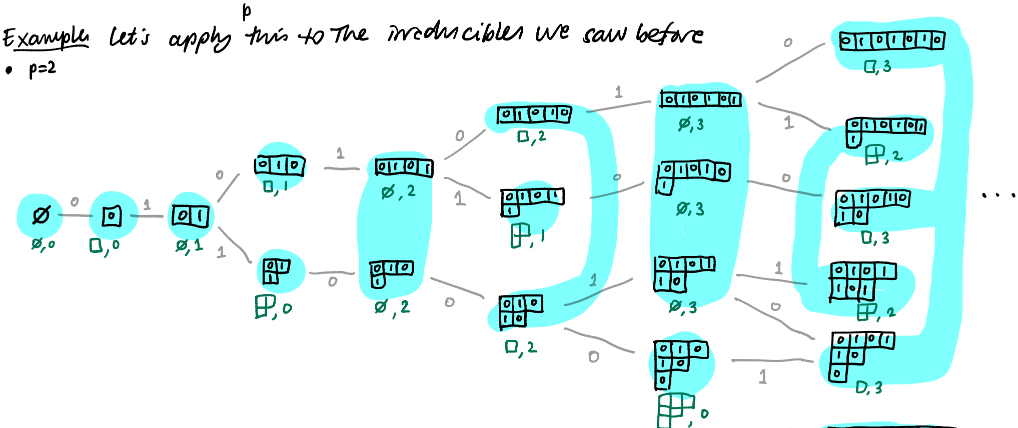
    ↑ they belong to a single block bc. they have block decomp but are irreducible

  Since n is fixed, we can talk about the <u>p-weight of a block</u>

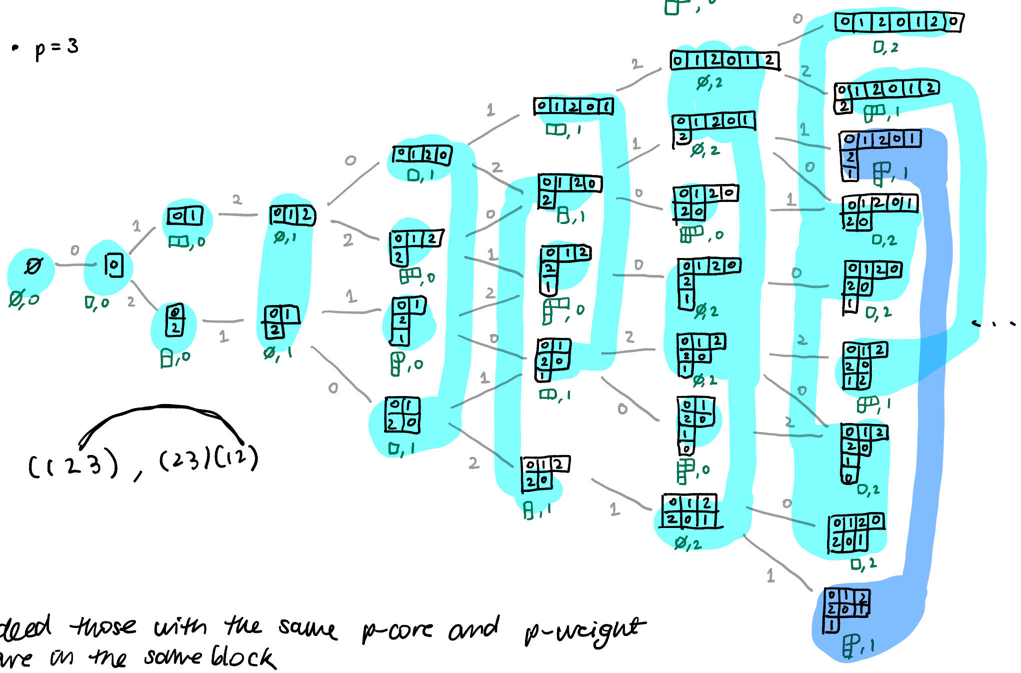
  ie. $p\text{-weight} = \dfrac{n - |p\text{-core}|}{p}$ by counting boxes removed from n to get p-core

  <u>Example</u> let's apply this to the irreducibles we saw before

  - $p=2$



  - $p = 3$



$(1\,2\,3)$, $(2\,3)(1\,2)$

Indeed those with the same p-core and p-weight are in the same block

<u>Remark</u> There is a more powerful equivalence when we categorify

$$\bigoplus_{n\geq 0} \mathrm{Rep}_p S_n = \bigoplus_{\substack{\text{weights} \\ \text{of } \widehat{sl_p}}} \mathcal{C}_\mu \qquad \text{block decomp}$$

then there is a derived equivalence

$$D^b(\mathcal{C}_\mu) \cong D^b(\mathcal{C}_{w\cdot\mu}) \quad , \forall w \in \widehat{sl_p}$$

[Analogue: Showing $V_n \cong V_{-n}$ for $\bigoplus_{i\in\mathbb{Z}} V_i$]

which implies Broué's Abelian defect conjecture for symmetric groups.

Proved by Chuang-Rouquier by an sl_2 action on $\bigoplus \mathrm{Rep}_p S_n$ and "integrate"