01/04/2026 Canonical bases

Outline

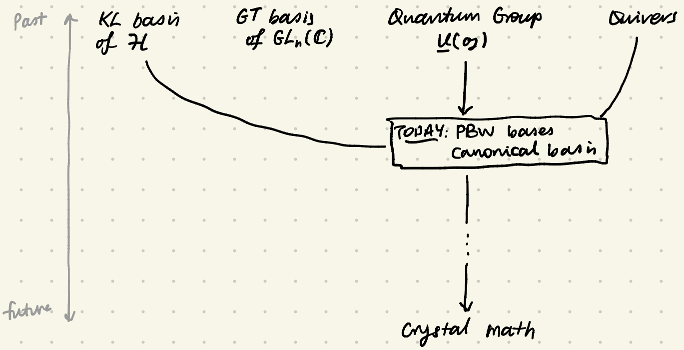
⓪ Recap and goal

① PBW basis for $U(\mathfrak{g})$ and $\underline{U}(\mathfrak{g})$

② Canonical basis for $\underline{U}(\mathfrak{g})$

⓪ GOAL Want

1) bases for irreps } canonical bases
2) characters
3) decompose tensors of irreps } crystal bases



Today we will build up to the canonical bases for $\underline{U}(\mathfrak{g})$ the quantum group

- see it as a generalisation of KL basis of $\mathcal{H}$
- we will use a little bit of quiver reps from last week
- Next time we should be equipped to answer (1)

① PBW bases for $U(\mathfrak{g})$ and $\underline{U}(\mathfrak{g})$

Let $\mathfrak{g}$ be simple Lie algebra $/\mathbb{C}$.

Φ the root system of $\mathfrak{g}$

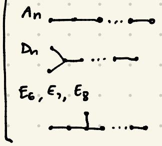
$\cup$

Φ^+ positive roots

$\cup$

$\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ a base of simple roots

Dynkin diagram



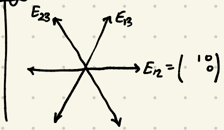
We restrict to simply laced case ie. $\mathfrak{g}$ of type ADE

Recall $U(\mathfrak{g}) = T(\mathfrak{g}) / [x,y] = xy - yx$ unital associative algebra

THM (Poincaré-Birkhoff-Witt theorem)

Given a total ordering of a basis in $\mathfrak{g}$, $(x_1, \ldots, x_N)$, the set of monomials
$$x_1^{p_1} x_2^{p_2} \ldots x_i^{p_i}$$
form a basis for $U(\mathfrak{g})$.

- often we take an ordering corresponding to the decomposition
$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$$
  - here, $\mathfrak{g}_+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ has a natural basis given by positive roots
  - $\boxed{\text{eg}}$ $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$



- Picking a total order on Φ^+ gives (via PBW) a basis for
$$U(\mathfrak{g}) = U^+ \otimes U^0 \otimes U^-$$
  - the basis for $\mathfrak{g}_+$ extends to a basis for U^+
  - $\boxed{\text{eg}}$ $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, $U^+(\mathfrak{g})$ has basis
$$\{E_{12}^{p_1} E_{23}^{p_2} E_{13}^{p_3} : p_1, p_2, p_3 \in \mathbb{Z}_{\geq 0}\}$$

The definition for $\underline{U}(\mathfrak{g})$ is given by generators indexed by simple roots $\alpha_i \in \Delta$

- this can also be done for $\mathfrak{g}$ and $U(\mathfrak{g})$, where the other positive roots come from $[-,-]$

  $\boxed{\text{eg}}$ $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ has simple roots E_{12}, E_{23}

  and $[E_{12}, E_{23}] = \begin{pmatrix} & 1 & \\ & & 0 \\ & & \end{pmatrix}\begin{pmatrix} & 0 & 0 \\ & & 1 \\ & & \end{pmatrix} - \begin{pmatrix} & 0 & 0 \\ & & 1 \\ & & \end{pmatrix}\begin{pmatrix} & 1 & 0 \\ & & 0 \\ & & \end{pmatrix} = \begin{pmatrix} & 0 & 1 \\ & & 0 \\ & & \end{pmatrix} = E_{13}$

  So in $U(\mathfrak{g})$,
$$E_{13} = E_{12} E_{23} - E_{23} E_{12}$$

- for $\underline{U}(\mathfrak{g})$ we don't already have E_{13} for free, so we use the serre relations to get them

  analogous to $[E_i, E_j]$ relations in $U(\mathfrak{g})$

Another simplification:

- We have $\underline{U} = \underline{U}^+ \otimes \underline{U}^0 \otimes \underline{U}^-$
  - $\underline{U}^+$: E's, $\underline{U}^0$: $K^{\pm}$'s, $\underline{U}^-$: F's
- $\underline{U}^0$ is commutative so has a basis of $K_i^{\pm}$ monomials (easy)
- Bases for $\underline{U}^+, \underline{U}^-$ can be obtained in similar ways

So we focus on the hardest part $\underline{U}^+$ (or alternatively $\underline{U}^-$)

- the relation between E_i's is what makes this hard

Recall from last lecture

FACT Let W Weyl group of $\mathfrak{g}$, $w_0 \in W$ longest element, take any $\underline{i} = (i_1, \dots, i_N)$ such that $s_{i_1} \cdots s_{i_N} = w_0$ is a reduced exp. Then every $\alpha \in \Phi^+$ appears exactly once in

$$\beta_1 = \alpha_{i_1}$$
$$\beta_2 = s_{i_1}(\alpha_{i_2})$$
$$\beta_3 = s_{i_1} s_{i_2}(\alpha_{i_3})$$
$$\vdots$$
$$\beta_N = s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{i_N})$$

- Write $\mathcal{I}$ for the set of such $\underline{i}$, "reduced expressions of w_0"
- We mimic this to construct "E_{β_j} for Φ^+" from "$E_i = E_{\alpha_i}$ for $\Delta \subset \Phi^+$". Then, like for $U(\mathfrak{g})$, ordered monomials will give a basis for $\underline{U}(\mathfrak{g})$.

First we need an analogue of the W-action for the E_i's.

one for each simple root α_i

FACT There are $\mathbb{Q}(v)$-algebra automorphisms $T_i^{-1} : \underline{U} \to \underline{U}$ for $i = 1, \dots, n$ given by

$$E_j \mapsto \begin{cases} -K_j^{-1} F_j & i=j \\ E_j & a_{ij}=0 \\ -E_j E_i + v^{-1} E_i E_j & a_{ij}=-1 \end{cases} \quad F_j \mapsto \begin{cases} -E_j K_j & i=j \\ F_j & a_{ij}=0 \\ -F_i F_j + v F_j F_i & a_{ij}=-1 \end{cases} \quad K_j = \begin{cases} K_j^{-1} & i=j \\ K_j & a_{ij}=0 \\ K_i K_j & a_{ij}=-1 \end{cases}$$

- Define $\mathbb{Q}(v)$-alg. automorphisms $r_i : \underline{U} \to \underline{U}$

$$E_j \mapsto (-1)^{a_{ij}} E_j$$
$$F_j \mapsto (-1)^{a_{ij}} F_j$$
$$K_j \mapsto K_j$$

and write $\tilde{T}_i := T_i^{-1} r_i$

FACT The morphisms $\tilde{T}_i$ for $i = 1, \dots, n$ satisfy

$$\tilde{T}_i \tilde{T}_j \tilde{T}_i = \tilde{T}_j \tilde{T}_i \tilde{T}_j \quad \text{if } a_{ij} = -1$$
$$\tilde{T}_i \tilde{T}_j = \tilde{T}_j \tilde{T}_i \quad \text{if } a_{ij} = 0$$

Also if $s_{i_1} \cdots s_{i_N} = w_0$ is a reduced word in W, and $s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_j$ then $\tilde{T}_{i_1} \cdots \tilde{T}_{i_{k-1}}(E_{i_k}) = E_j$

This defines the action of a <u>braid group</u> $B_{\mathfrak{g}} = \langle \tilde{T}_1, \dots, \tilde{T}_n \mid \cdots \rangle$ on $\underline{U}(\mathfrak{g})$

DEF Let $\underline{i} = (i_1, \dots, i_N) \in \mathcal{I}$, so that $\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_k)$ for $k = 1, \dots, N$.

Write $E_{\beta_k}^{(r)} := \tilde{T}_{i_1} \tilde{T}_{i_2} \cdots \tilde{T}_{i_{k-1}}(E_k^{(r)})$ (one for each root in Φ^+)

recall $E_k^{(r)} := \dfrac{E_k^r}{[r]!}$

Define $E_{\underline{i}}^{\underline{c}} := E_{\beta_1}^{(c_1)} E_{\beta_2}^{(c_2)} \cdots E_{\beta_N}^{(c_N)}$ when β_k depend on $\underline{i}$ as above and $\underline{c} \in \mathbb{N}^N$.

Fixing a choice of $\underline{i} \in \mathcal{I}$,

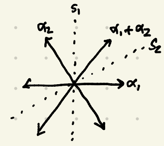
PROP The set $B_{\underline{i}} = \{E_{\underline{i}}^{\underline{c}} : \underline{c} \in \mathbb{N}^N\}$ is a $\mathbb{Q}(v)$ basis for $\underline{U}^+$ and is $\underline{U}_{\mathbb{Z}}^+$.

This is the PBW basis for $\underline{U}^+(\mathfrak{g})$ and extends to a basis for $\underline{U}(\mathfrak{g})$.
- Depends on choice of $\underline{i}$ ie. reduced word of w_0 ... highly non-canonical!

↳ for type A, [Stanley 1984], OEIS A005118, is $O(n^{n^2/2})$ growth

eg $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$

- $\underline{U}^+$ is generated by $E_1 = E_{\alpha_1}$, $E_2 = E_{\alpha_2}$
- Take $\underline{i} = (1,2,1)$ for $s_1 s_2 s_1 = w_0$ in $W = S_3$



$\beta_1 = \alpha_1$ | $E_{\beta_1} = E_1$

$\beta_2 = s_1(\alpha_2) = \alpha_1+\alpha_2$ and $E_{\beta_2} = \tilde{T}_1(E_2) = E_2 E_1 - v^{-1} E_1 E_2$

$\beta_3 = s_1 s_2(\alpha_1) = \alpha_2$ | $E_{\beta_3} = \tilde{T}_1 \tilde{T}_2 (E_1) = \tilde{T}_1(E_1E_2 - v^{-1}E_2E_1)$

$$= (-K_1^{-1}F_1)(E_2E_1 - v^{-1}E_1E_2) - v^{-1}(E_2E_1 - v^{-1}E_1E_2)(-K_1^{-1}F_1)$$
$$= \cdots$$
$$= E_2$$

(ex)

② Canonical basis for $\underline{U}(\mathfrak{g})$

We want to remove this dependence on $\underline{i}$.
Given two $\underline{i}, \underline{i}' \in \mathcal{I}$, we can write
$$E_{\underline{i}}^{\underline{c}} = \sum_{\underline{c}' \in \mathbb{N}^N} \gamma_{\underline{i},\underline{i}'}^{\underline{c},\underline{c}'} E_{\underline{i}'}^{\underline{c}'}$$
for $\gamma_{\underline{i},\underline{i}'}^{\underline{c},\underline{c}'} \in \mathbb{Q}(v)$.

PROP (Lusztig, 2.3) let $\underline{i} \in \mathcal{I}$.

ⓐ $B_{\underline{i}}$ is a basis of $\underline{U}_{\mathbb{Z}}^+$ as a $\mathbb{Z}[v^{\pm1}]$-module

ⓑ If $\mathcal{L}_{\underline{i}}$ is the $\mathbb{Z}[v^{-1}]$-submodule of $\underline{U}^+$ generated by $B_{\underline{i}}$, then $\mathcal{L}_{\underline{i}}$ is indep. of $\underline{i}$. We call it $\mathcal{L}$.

ⓒ Let $\pi: \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ be the obvious projection. Then $\pi(B_{\underline{i}})$ is a $\mathbb{Z}$-basis of $\mathcal{L}/v^{-1}\mathcal{L}$ and is indep. of $\underline{i}$. We call it B.

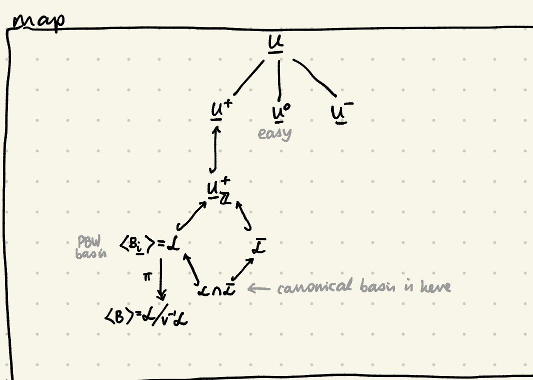
Proof idea
- By Matsumoto, any $\underline{i}, \underline{i}' \in \mathcal{I}$ are related by braid relations.
- If $\underline{i}, \underline{i}'$ is related by one braid relation, say $\underline{i} = (1,2,1)$ $\underline{i}' = (2,1,2)$
  then define $R_{\underline{i}}^{\underline{i}'}: \underline{c} = (x,y,z) \longmapsto (y+z-\min(x,z),\ \min(x,z),\ x+y-\min(x,z))$.
- Using the relations in $\underline{U}$, it can be shown that
$$\gamma_{\underline{i},\underline{i}'}^{\underline{c},\underline{c}'} \in \begin{cases} 1 + v^{-1}\mathbb{Z}[v^{-1}] & \text{if } R_{\underline{i}}^{\underline{i}'}\underline{c} = \underline{c}' \\ v^{-1}\mathbb{Z}[v^{-1}] & \text{otherwise} \end{cases}$$
- Inductively we can extend this to any $\underline{i}, \underline{i}'$.
- ⓐ, ⓑ, ⓒ are a consequence of this $\square$

For us, we want to focus on ⓑ, ⓒ because they give a glimpse of something canonical

DEF Let $\bar{\ }: \underline{U} \to \underline{U}$ be the $\mathbb{Q}$-alg morphism, called "bar involution"
$$E_i \longmapsto E_i$$
$$F_i \longmapsto F_i$$
$$K_i \longmapsto K_i^{-1}$$
$$v \longmapsto v^{-1}$$

- Apply $\bar{\ }$ to $\mathcal{L}$ to get a $\mathbb{Z}[v]$-submodule $\bar{\mathcal{L}}$ of $\underline{U}_{\mathbb{Z}}^+$
- Want to mimic KL-basis for $\mathcal{H}$: self dual & unitriangular

map



* THM (Lusztig 3.2)

ⓐ The morphism of $\mathbb{Z}$-modules
$$\pi': \mathcal{L} \cap \bar{\mathcal{L}} \hookrightarrow \mathcal{L} \xrightarrow{\pi} \mathcal{L}/v^{-1}\mathcal{L}$$
is an isomorphism.

hidden in def π, B is unitriangularity

ⓑ $\underline{B} = (\pi')^{-1}(B)$ is a $\mathbb{Z}$-basis of $\mathcal{L} \cap \bar{\mathcal{L}}$
a $\mathbb{Z}[v^{-1}]$-basis of $\mathcal{L}$
a $\mathbb{Z}[v]$-basis of $\bar{\mathcal{L}}$
a $\mathbb{Z}[v^{\pm1}]$-basis of $\underline{U}_{\mathbb{Z}}^+$
a $\mathbb{Q}(v)$-basis of $\underline{U}^+$.

ⓒ Every element $b \in \underline{B}$ satisfies $\bar{b} = b$.

The canonical basis element $\mathcal{E}^{\underline{c}}$ is unique such that

① $\pi(\mathcal{E}^{\underline{c}}) = \pi(E_{\underline{i}}^{\underline{c}})$

② $\overline{\mathcal{E}^{\underline{c}}} = \mathcal{E}^{\underline{c}}$

— halfway mark

Examples

- A_1, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ the canonical basis is the same as PBW basis
$$\{E_1^{(m)} : m \in \mathbb{N}\}$$

- A_2, $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ the canonical basis is
$$\{E_1^{(m)} E_2^{(m')} E_3^{(m'')} : m, m', m'' \in \mathbb{N},\ m' \geq m + m''\} \cup \{E_2^{(m)} E_1^{(m')} E_2^{(m'')} : m, m', m'' \in \mathbb{N},\ m' \geq m + m''\}$$
where $E_1^{(m)} E_2^{(m+m'')} E_1^{(m'')} = E_2^{(m'')} E_1^{(m+m'')} E_2^{(m)}$ is where they overlap

  - compare w/ PBW basis for $\underline{i} = (1,2,1)$: $E_{\beta_1}^{(c_1)} E_{\beta_2}^{(c_2)} E_{\beta_3}^{(c_3)}$ where $E_{\beta_2} = E_2 E_1 - v^{-1} E_1 E_2$

- A_3, $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ the canonical basis contains
$$E_2^{(2)} E_1 E_3 E_2 - E_2^{(3)} E_1 E_3 = E_2 E_1 E_3 E_2^{(2)} - E_1 E_3 E_2^{(2)}$$
not a monomial

As with the KL basis, the hard part is showing existence. This is constructed inductively from any PBW basis and uni-triangular wrt. an order we saw in Joe's talk...

- let Ω be an ADE quiver. Fix $\mathbb{F} = \mathbb{F}_q$, q = prime power
- Fix $\underline{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$. Then $\underline{E}_{\underline{d}} \curvearrowleft G_{\underline{d}}$ such that
$$\text{same } G_{\underline{d}}\text{-orbit} \iff \text{isomorphic } \Omega\text{-modules}$$

- We may write β_k in terms of simple roots α_j:
$$\beta_k = \sum_{j=1}^{n} p_j^k \alpha_j \quad , \quad p_j^k \in \mathbb{N}$$

Define $\chi_{\underline{i}} : \mathbb{N}^N \to \mathbb{N}^n$ that sends $\underline{c} = (c_1, \dots, c_N) \longmapsto \underline{d} = (d_1, \dots, d_n)$ such that $d_j = \sum_{k=1}^{N} p_j^k c_k$.
– The fibres are finite: $\chi_{\underline{i}}^{-1}(\underline{d})$ indexes all Ω-mod$_q$/iso with dim $= \underline{d}$ (or equiv, orbits $\underline{E}_{\underline{d}}/G_{\underline{d}}$)

- Given $\underline{i} \in \mathcal{I}$, $\underline{c} \in \chi_{\underline{i}}^{-1}(\underline{d})$, we can define a Ω-module $V_{\underline{c}}$, w/ $\dim V_{\underline{c}} = \underline{d}$ and orbit $\mathcal{O}_{\underline{c}}$

DEF For $\underline{c}, \underline{c}' \in \chi_{\underline{i}}^{-1}(\underline{d})$ we say $\underline{c} \leq \underline{c}'$ if $\underline{c} = \underline{c}'$ or $\dim \mathcal{O}_{\underline{c}} < \dim \mathcal{O}_{\underline{c}'}$.
- Later we may see we can define this partial order via orbit closures like for Schubert varieties
- This is the order by which the canonical basis is unitriangular to PBW bases.

Proof outline

- inclusion $\Gamma : R\Omega \longrightarrow \mathcal{U}_q$ ⇝ formula for products in $\underline{\mathcal{U}}$ via quiver representations
- applying Γ, we deduce the uni-triangularity of $\bar{\ }$ in PBW basis wrt. $\leq$
- generalised KL polynomials ⇝ canonical bases, uni-triangular to PBW wrt. $\leq$

quivers!

Recall, with $q \in \mathbb{C}$ $\quad \mathbb{Z}[v^{\pm 1}] \curvearrowright \mathbb{C}$ $\quad (v \mapsto q^{\frac{1}{2}})$, we extend $\quad U_q := \underline{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}[v^{\pm1}]} \mathbb{C}$

- call U_q^0, U_q^+ subalgebras gen. by K's and E's respectively $\quad$ and write $U_q^{\geq 0} := U_q^0 \otimes_{\mathbb{C}} U_q^+$

We have the magic isomorphism of $\mathbb{C}$-algebras

$$U_q^{\geq 0} \xrightarrow{\sim} R\Omega \left(\simeq U_q^0 \otimes_{\mathbb{C}} R\Omega\right)$$

$$K_i^{\pm 1} \longmapsto K_i^{\pm 1}$$

$$E_i \longmapsto K_i^{z_i} E_i$$

integers coming from Ω (not unique)

Taking the inverse and restricting to $R\Omega$, we have an inclusion

$$\Gamma : R\Omega \hookrightarrow U_q$$

$$V_{\underline{c}} \longmapsto q^{f_{\underline{c}}/2} K E_{\underline{i}}^{\underline{c}} \qquad \text{for } \underline{c} \in \mathbb{N}^N$$

for $f_{\underline{c}} \in \mathbb{Z}$, K monomial in $K_i^{\pm}$ dep. on $\underline{d} = \dim V_{\underline{c}}$

- We want to use product in $R\Omega$ to deduce an equation in U_q.

Let $V \in \mathrm{Mod}_q \Omega$ w/ $\dim V = \underline{d}$ and $V_i \in \mathrm{Mod}_q \Omega$ w/ i^{th} comp as in V and others 0.
By def of $R\Omega$, $\quad V_1 \cdot V_2 = \sum_W g_{W,V_1,V_2} W \quad$ where $g_{W,V_1,V_2} = \#\ \varphi$ s.t. $\exists$ exact $0 \to V_1 \xrightarrow{\varphi} W \to V_2 \to 0$

$$V_n \cdots V_2 V_1 = \sum_{\underline{c} \in X_{\underline{i}}^{-1}(\underline{d})} V_{\underline{c}}$$

$$\Gamma(V_n \cdots V_2 V_1) = \sum_{\underline{c}} \Gamma(V_{\underline{c}}) \qquad \text{in } U_q$$

$$\overset{\text{some manip.}}{\rightsquigarrow} \quad E_n^{(d_n)} \cdots E_1^{(d_1)} = \sum_{\underline{c}} v^{f_{\underline{c}} - r_{\underline{d}}} E_{\underline{i}}^{\underline{c}} \qquad \text{in } \underline{U}_{\mathbb{Z}}$$

PROP (Lusztig 7.7)
We have

$$E_n^{(d_n)} E_{n-1}^{(d_{n-1})} \cdots E_1^{(d_1)} = \sum_{\underline{c} \in X_{\underline{i}}^{-1}(\underline{d})} v^{f_{\underline{c}} - r_{\underline{d}}} E_{\underline{i}}^{\underline{c}}$$

a number dep. on $\underline{d}$

where $f_{\underline{c}} - r_{\underline{d}} = -\delta(\underline{c}) = -\mathrm{codim}\ \mathcal{O}_{\underline{c}}$

- Lusztig spends the start of sec.7 calculating $f_{\underline{c}}$

These are analogous to R-polys for KL bases

- A consequence of this is that we can write

$$\overline{E_{\underline{i}}^{\underline{c}}} = E_{\underline{i}}^{\underline{c}} + \sum_{\substack{\underline{c}' \in X_{\underline{i}}^{-1}(\underline{d}) \\ \underline{c}' < \underline{c}}} \omega_{\underline{c}'}^{\underline{c}} E_{\underline{i}}^{\underline{c}'} \qquad \text{for } \omega_{\underline{c}'}^{\underline{c}} \in \mathbb{Z}[v^{\pm 1}]$$

- i.e. the $\bar{\cdot}$ involution is unitriangular wrt. "$\leq$" order on $X_{\underline{i}}^{-1}(\underline{d})$
- this is key to constructing the canonical basis

- A technical lemma

LEM Let I be poset s.t. $\forall\, i, i' \in I$, $\exists\, u_{i'}^i \in \mathbb{Z}[v^{\pm 1}]$ s.t.

ⓐ $\displaystyle\sum_{i'' \in I} u_{i'}^{i''} \overline{u_{i''}^{i}} = \delta_{i,i'} \qquad \forall\, i, i' \in I$

ⓑ $u_i^i = 1 \qquad \forall\, i \in I$

ⓒ $u_i^{i'} = 0 \qquad$ unless $i' \leq i$

Then the system of equations $\left\{ z_{i'}^i = \sum_{i'' : i' \leq i'' \leq i} u_{i'}^{i''} \overline{z_{i''}^{i}} \;\middle|\; i' \leq i \right\}$ in variables $z_{i'}^i \in \mathbb{Z}[v^{-1}]$, $i' \leq i$,
has a unique solution s.t.
- $z_i^i = 1 \quad \forall\, i \in I$
- $z_{i'}^i \in v^{-1}\mathbb{Z}[v^{-1}]$ for $i' < i$

this lemma is exactly how KL-polys come from R-polys

Proof Almost exactly the same as proof for KL basis of Hecke algebra
- given inductively

Now for the construction.

- For us, fix $\underline{i} \in \mathcal{I}$ and $\underline{d} \in \mathbb{N}^n$. The lemma gives us change of basis coeffs from the PBW basis for $\underline{i}$

In the lemma, take
  - $I = \chi_{\underline{i}}^{-1}(\underline{d})$ with partial order $\underline{c} \le \underline{c}'$ defined above.
  - $u_{i'}^{i}$ to be $\omega_{\underline{c}'}^{\underline{c}}$

(ex)
- This satisfies ⓐ as

$$E_{\underline{i}}^{\underline{c}} = \sum_{\underline{c}'} \overline{\omega_{\underline{c}'}^{\underline{c}}}\, \overline{E_{\underline{i}}^{\underline{c}'}}$$

$$= \sum_{\underline{c}'} \overline{\omega_{\underline{c}'}^{\underline{c}}} \Big(\sum_{\underline{c}''} \omega_{\underline{c}''}^{\underline{c}'} E_{\underline{i}}^{\underline{c}''} \Big)$$

So coefficients (as a basis) gives

$$\sum_{\underline{c}'} \omega_{\underline{c}''}^{\underline{c}'} \overline{\omega_{\underline{c}'}^{\underline{c}}} = \begin{cases} 1 & \text{if } \underline{c}'' = \underline{c} \\ 0 & \text{otherwise} \end{cases}$$

- ⓑ, ⓒ are satisfied by previous PROP.

Hence we get a unique solution to the system of equations

$$\Big\{ \zeta_{\underline{c}'}^{\underline{c}} = \sum_{\underline{c}'' : \underline{c}' \le \underline{c}'' \le \underline{c}} \omega_{\underline{c}'}^{\underline{c}''} \overline{\zeta_{\underline{c}''}^{\underline{c}}} \;\Big|\; \underline{c}' \le \underline{c} \Big\} \quad \text{in variables } \zeta_{\underline{c}'}^{\underline{c}} \in \mathbb{Z}[v^{-1}] \text{ for } \underline{c}' \le \underline{c}'$$

- With $\zeta_{\underline{c}'}^{\underline{c}}$ unitriangular and lower terms in $v^{-1}\mathbb{Z}[v^{-1}]$
- These generalise KL-polys

- For $\underline{c} \in I$, let $\mathcal{E}^{\underline{c}} = E_{\underline{i}}^{\underline{c}} + \sum_{\underline{c}' < \underline{c}} \zeta_{\underline{c}'}^{\underline{c}} E_{\underline{i}}^{\underline{c}'}$ and $\underline{B} = \{ \mathcal{E}^{\underline{c}} : \underline{c} \in \chi_{\underline{i}}^{-1}(\underline{d}),\ \underline{d} \in \mathbb{N}^n \}$. This is the canonical basis!

note this

Let's check the construction satisfies our needs
- The set $\underline{B}$ has our desired properties
  - by the lemma, the coefficients are unitriangular to the PBW basis for $\underline{i}$.
  - this is self dual by unitriangularity of $\zeta_{\underline{c}'}^{\underline{c}}$ and $\bar{\cdot}$ on PBW basis

(ex)
$$\overline{\mathcal{E}^{\underline{c}}} = \sum_{\underline{c}' \le \underline{c}} \overline{\zeta_{\underline{c}'}^{\underline{c}}}\, \overline{E_{\underline{i}}^{\underline{c}'}} = \sum_{\underline{c}' \le \underline{c}} \overline{\zeta_{\underline{c}'}^{\underline{c}}} \Big(\sum_{\underline{c}'' \in I} \omega_{\underline{c}''}^{\underline{c}'} E_{\underline{i}}^{\underline{c}''} \Big) = \sum_{\underline{c}'' \in I} \Big(\sum_{\underline{c}' \le \underline{c}} \overline{\zeta_{\underline{c}'}^{\underline{c}}} \omega_{\underline{c}''}^{\underline{c}'} \Big) E_{\underline{i}}^{\underline{c}''} = \sum_{\underline{c}'' \le \underline{c}} \zeta_{\underline{c}''}^{\underline{c}} E_{\underline{i}}^{\underline{c}''}$$

- It is indeed a basis for $U_{\mathbb{Z}}^+$, U^+, and $\mathcal{L}$ over the appropriate rings bc it is unitriangular to PBW basis
- It doesn't depend on the PBW basis we started from:
  - $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ has

bc. other terms have coeff in $v^{-1}\mathbb{Z}[v^{-1}]$

$$\pi(\mathcal{E}^{\underline{c}}) = \pi(E_{\underline{i}}^{\underline{c}}) \quad \text{for all } \underline{c} \in I$$

so π is a bijection of $\underline{B} \subseteq \mathcal{L}$ and $B \subseteq \mathcal{L}/v^{-1}\mathcal{L}$
  - $\underline{B}$ is self dual so it is a $\mathbb{Z}$-basis for $\mathcal{L} \cap \bar{\mathcal{L}}$.

$$\Rightarrow \pi' : \mathcal{L} \cap \bar{\mathcal{L}} \xrightarrow{\sim} \mathcal{L}/v^{-1}\mathcal{L} \text{ is an iso}$$

  - Hence $\underline{B} = (\pi')^{-1}(B)$, so it doesn't depend on $\underline{i}$ since B doesn't

Remark Some comparisons to the KL-basis of $\mathcal{H}(W)$

$\underline{U}(\mathfrak{g})$	$\mathcal{H}(W)$
PBW highly non-canonical	standard basis δ_w is unique
change of basis given by IC sheaves of quiver varieties $\overline{\mathcal{O}_{\underline{c}}}$	KL polys given by dims of IC sheaves on Schubert var $\overline{X_w}$
change of basis is unitriangular to all PBW bases	change of basis is unitriangular

change of basis coefficients are constructed similarly (via induction)

We also get this canonical basis for $\underline{U}(\mathfrak{g})$-modules ... Jensen will talk about them next time