The Diagrammatic Lusztig–Vogan Category

for $\mathrm{SL}(2, \mathbb{R})$

Victor Zhang

Supervised by
Dr Anna Romanov
A/Prof Pinhas Grossman

UNSW Sydney

November 2024

Outline

1. Context and Motivation
2. Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$
3. Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2, \mathbb{R})$
4. Further work

Context and Motivation

Complex (simply-connected) Lie groups

Context and Motivation

Complex (simply-connected) Lie groups

BGG
Category $\mathcal{O}$

Context and Motivation

Complex (simply-connected) Lie groups

$$\begin{tikzcd}
\begin{array}{c}\text{BGG}\\ \text{Category } \mathcal{O}\end{array} \arrow[r, squiggly] & \begin{array}{c}\text{Hecke}\\ \text{algebra } \mathcal{H}\end{array}
\end{tikzcd}$$

Context and Motivation

Complex (simply-connected) Lie groups

$$\begin{tikzcd}
\text{BGG Category } \mathcal{O} \arrow[r, squiggly] & \text{Hecke algebra } \mathcal{H} \arrow[r, squiggly, "\text{categori-fication}"] & \text{Soergel bimodules } \mathbb{S}\text{Bim}
\end{tikzcd}$$

Context and Motivation

Complex (simply-connected) Lie groups

$$\begin{tikzcd}
\begin{array}{c}\text{BGG}\\ \text{Category } \mathcal{O}\end{array} \arrow[r, rightsquigarrow] & \begin{array}{c}\text{Hecke}\\ \text{algebra } \mathcal{H}\end{array} \arrow[r, rightsquigarrow, "\begin{array}{c}\text{categori-}\\ \text{fication}\end{array}"] & \begin{array}{c}\text{Soergel}\\ \text{bimodules}\\ \mathbb{S}\text{Bim}\end{array} \arrow[r, rightsquigarrow] & \begin{array}{c}\text{Diagrammatics}\\ \mathcal{D}\,\mathbb{S}\text{Bim}\end{array}
\end{tikzcd}$$

Context and Motivation

Complex (simply-connected) Lie groups

$$\begin{tikzcd} \text{BGG Category } \mathcal{O} \arrow[r, rightsquigarrow] & \text{Hecke algebra } \mathcal{H} \arrow[r, rightsquigarrow, "\text{categori-fication}"] & \text{Soergel bimodules } \mathbb{S}\text{Bim} \arrow[r, rightsquigarrow] & \text{Diagrammatics } \mathcal{D}\,\mathbb{S}\text{Bim} \end{tikzcd}$$

Real (reductive) Lie groups

Context and Motivation

Complex (simply-connected) Lie groups

$$\begin{tikzcd} \text{BGG Category } \mathcal{O} \arrow[r, rightsquigarrow] & \text{Hecke algebra } \mathcal{H} \arrow[r, rightsquigarrow, "\text{categori-fication}"] & \text{Soergel bimodules } \mathbb{S}\text{Bim} \arrow[r, rightsquigarrow] & \text{Diagrammatics } \mathcal{D}\,\mathbb{S}\text{Bim} \end{tikzcd}$$

Real (reductive) Lie groups

Category of
admissible reps.

Context and Motivation

Complex (simply-connected) Lie groups

$$\begin{tikzcd}
\text{BGG Category } \mathcal{O} \arrow[r, squiggly] & \text{Hecke algebra } \mathcal{H} \arrow[r, squiggly, "\text{categori-fication}"] & \text{Soergel bimodules } \mathbb{S}\mathrm{Bim} \arrow[r, squiggly] & \text{Diagrammatics } \mathcal{D}\,\mathbb{S}\mathrm{Bim}
\end{tikzcd}$$

Real (reductive) Lie groups

$$\begin{tikzcd}
\text{Category of admissible reps.} \arrow[r, squiggly] & \text{LV-module } M \curvearrowleft \mathcal{H}
\end{tikzcd}$$

Context and Motivation

Complex (simply-connected) Lie groups

$$\begin{tikzcd}
\text{BGG Category } \mathcal{O} \arrow[r, rightsquigarrow] & \text{Hecke algebra } \mathcal{H} \arrow[r, rightsquigarrow, "\text{categorification}"] & \text{Soergel bimodules } \mathbb{S}\mathrm{Bim} \arrow[r, rightsquigarrow] & \text{Diagrammatics } \mathcal{D}\,\mathbb{S}\mathrm{Bim}
\end{tikzcd}$$

Real (reductive) Lie groups

$$\begin{tikzcd}
\text{Category of admissible reps.} \arrow[r, rightsquigarrow] & \text{LV-module } M \curvearrowleft \mathcal{H} \arrow[r, rightsquigarrow, "\text{categorification}"] & \text{LV-category } \mathcal{N}^0 \curvearrowleft \mathbb{S}\mathrm{Bim}
\end{tikzcd}$$

Context and Motivation

Complex (simply-connected) Lie groups

$$\begin{tikzcd}
\text{BGG Category } \mathcal{O} \arrow[r, rightsquigarrow] & \text{Hecke algebra } \mathcal{H} \arrow[r, rightsquigarrow, "\text{categorification}"] & \text{Soergel bimodules } \mathbb{S}\mathrm{Bim} \arrow[r, rightsquigarrow] & \text{Diagrammatics } \mathcal{D}\,\mathbb{S}\mathrm{Bim}
\end{tikzcd}$$

Real (reductive) Lie groups

$$\begin{tikzcd}
\text{Category of admissible reps.} \arrow[r, rightsquigarrow] & \text{LV-module } M \curvearrowleft \mathcal{H} \arrow[r, rightsquigarrow, "\text{categorification}"] & \text{LV-category } \mathcal{N}^0 \curvearrowleft \mathbb{S}\mathrm{Bim} \arrow[r, rightsquigarrow, "?"] & \text{Diagrammatics } \mathcal{D}\mathcal{N}^0 \curvearrowleft \mathcal{D}\,\mathbb{S}\mathrm{Bim}
\end{tikzcd}$$

Context and Motivation

Why diagrammatics?

- Diagrammatic methods enabled Elias & Williamson (2014) to give an algebraic proof of the Kazhdan–Lusztig conjecture (1979). This is more general than can be proved geometrically.
- Provided intuition for Williamson (2017) to discover counterexamples to Lusztig's conjecture (1980).
- $\mathcal{D}\,\mathbb{S}\mathrm{Bim}$ can be studied in contexts where $\mathbb{S}\mathrm{Bim}$ is not well-behaved, for example in fields of characteristic p.

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

The diagrammatic Hecke category $\mathcal{DH}$ (for $\mathfrak{sl}_2\mathbb{C}$) is a $\mathbb{Z}$-linear monoidal category defined as follows.

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

The diagrammatic Hecke category $\mathcal{DH}$ (for $\mathfrak{sl}_2\mathbb{C}$) is a $\mathbb{Z}$-linear monoidal category defined as follows.

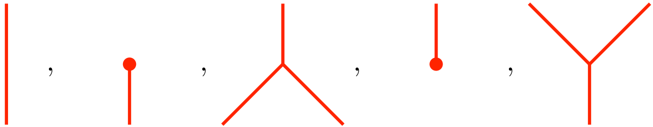
- *Objects*: generated by ●
  i.e. $\mathbb{1}$, ●, ●● $:=$ ● $\otimes$ ●, ●●●, ●●●●, ...

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

The diagrammatic Hecke category $\mathcal{DH}$ (for $\mathfrak{sl}_2\mathbb{C}$) is a $\mathbb{Z}$-linear monoidal category defined as follows.

- *Objects*: generated by ●
  i.e. $\mathbb{1}$, ●, ●● $:=$ ● $\otimes$ ●, ●●●, ●●●●, ...
- *Morphisms*: generated by

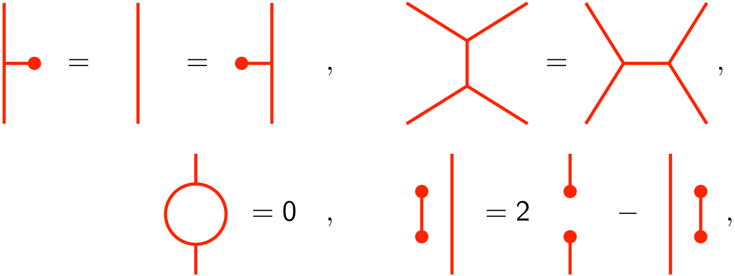


under relations...

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

Relations

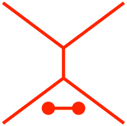


and arbitrary planar isotopy.

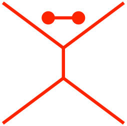
Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$\frac{1}{2}$



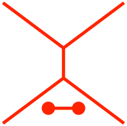
$+\frac{1}{2}$



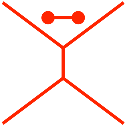
Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

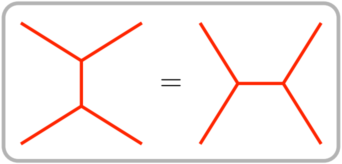
Example

$\frac{1}{2}$



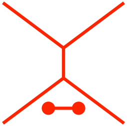
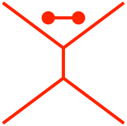
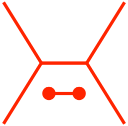
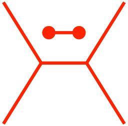
$+\ \frac{1}{2}$

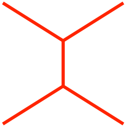
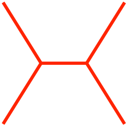




Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

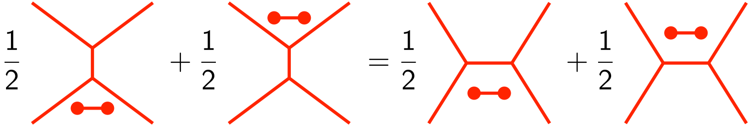
Example

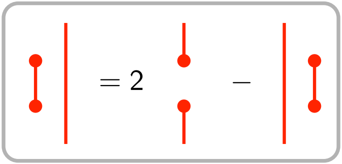
$\frac{1}{2}$  $+ \frac{1}{2}$  $= \frac{1}{2}$  $+ \frac{1}{2}$ 

 $=$ 

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

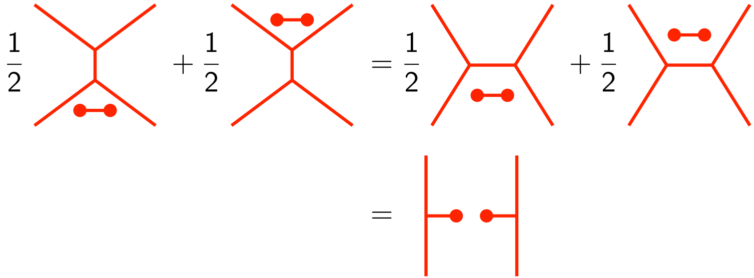
Example

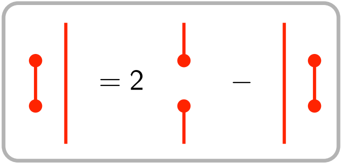




Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

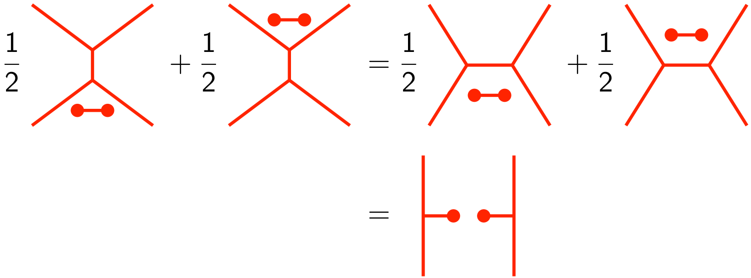
Example

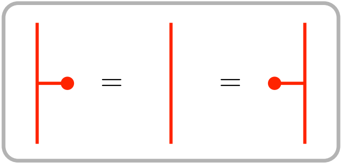




Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

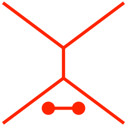
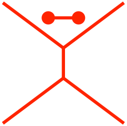
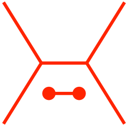
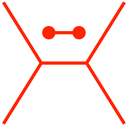
Example

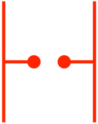
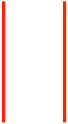




Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$\frac{1}{2}$  $+\ \frac{1}{2}$  $=\ \frac{1}{2}$  $+\ \frac{1}{2}$ 

$=$  $=$ 

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Theorem (Elias–Khovanov, 2010[1])

The Karoubi envelope of $\mathcal{DH}$ *is equivalent to the category of Soergel Bimodules* $\mathbb{S}\mathrm{Bim}$ *for* $\mathfrak{sl}_2(\mathbb{C})$ *as graded additive* $\mathbb{R}$*-linear monoidal categories.*

---

[1] Ben Elias and Mikhail Khovanov. "Diagrammatics for Soergel categories". In: *Int. J. Math. Math. Sci.* (2010), Art. ID 978635, 58.

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Definition

The diagrammatic LV-category $\mathcal{D}\tilde{\mathcal{N}}^0$ (for $\mathrm{SL}(2,\mathbb{R})$) is a $\mathbb{Z}$-linear right module category over $\mathcal{DH}$ defined as follows. The right action by $\mathcal{DH}$ is right concatenation.

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Definition

The diagrammatic LV-category $\mathcal{D}\tilde{\mathcal{N}}^0$ (for $\mathrm{SL}(2,\mathbb{R})$) is a $\mathbb{Z}$-linear right module category over $\mathcal{DH}$ defined as follows. The right action by $\mathcal{DH}$ is right concatenation.

- *Objects*: generated by $\mathbb{1}$ and ○
  i.e. $\mathbb{1}$, ●, ●●, ●●●, ●●●●, ... and ○, ○●, ○●●, ...

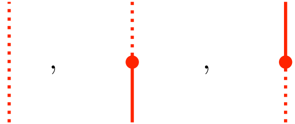
Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Definition

The diagrammatic LV-category $\mathcal{D}\tilde{\mathcal{N}}^0$ (for $\mathrm{SL}(2,\mathbb{R})$) is a $\mathbb{Z}$-linear right module category over $\mathcal{DH}$ defined as follows. The right action by $\mathcal{DH}$ is right concatenation.

- *Objects*: generated by $\mathbb{1}$ and ○
  i.e. $\mathbb{1}$, ●, ●●, ●●●, ●●●●, ... and ○, ○●, ○●●, ...
- *Morphisms*: generated by

  $\mathrm{id}_{\mathbb{1}}$,

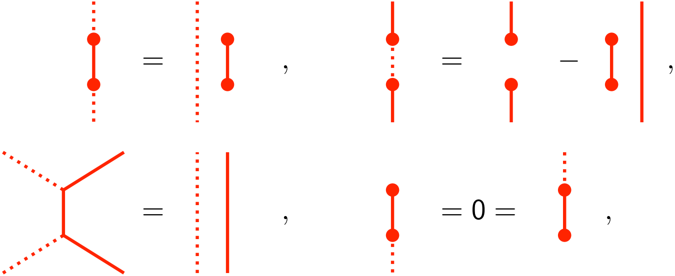


  under relations...

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Definition

Relations



and arbitrary planar isotopy while dotted red lines never appear right of any red.

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Definition

Remark

This diagrammatic definition is not entirely new: Elias–Williamson[2] had constructed very similar diagrammatics for localisation of Soergel bimodules. The LV-category is a subcategory of this, with restrained objects, morphisms and relations.

---

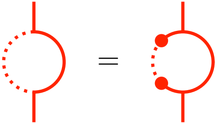
[2] Ben Elias and Geordie Williamson. “Soergel calculus”. In: *Represent. Theory* 20 (2016), pp. 295–374.

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2, \mathbb{R})$

Example

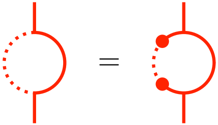
Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2, \mathbb{R})$

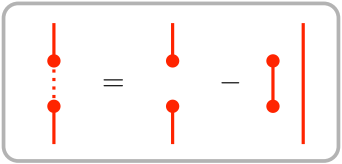
Example



Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2, \mathbb{R})$

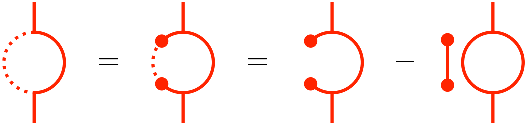
Example

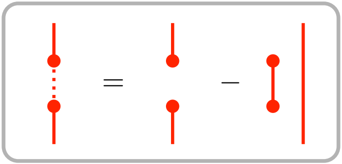




Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

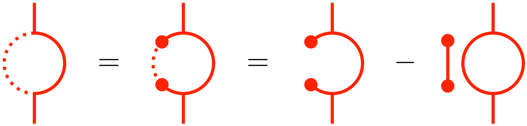
Example





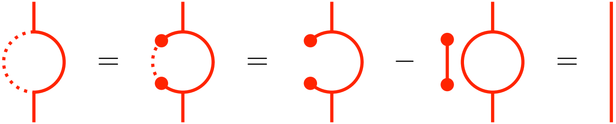
Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Example



Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Example



Diagrammatic Lusztig–Vogan category for $\mathsf{SL}(2,\mathbb{R})$

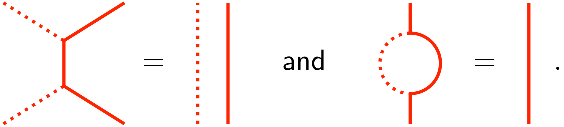
Theorem (Z. 2024)

The Karoubi envelope of $\mathcal{D}\tilde{\mathcal{N}}^0$ *is equivalent to the Lusztig–Vogan category* $\mathcal{N}^0$ *for* $\mathsf{SL}(2,\mathbb{R})$ *as graded additive* $\mathbb{R}$*-linear right module categories over* $\mathbb{S}\mathsf{Bim}$ *(for* $\mathfrak{sl}_2\mathbb{C}$*).*

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Proof idea

The backbone of the proof is the isomorphism ● $\simeq$ ○●, given by the relations



This reflects an isomorphism on the level of modules that allows us to construct a basis of morphisms from an existing one in $\mathcal{DH}$.

Further Work

- There is an isomorphism $\mathrm{SL}(2, \mathbb{R}) \simeq \mathrm{SU}(1, 1)$. A natural next step is to consider $\mathrm{SU}(2, 1)$, with an aim to define diagrammatics for the infinite family $\mathrm{SU}(n, 1)$.
- Understand the dimension of the general morphism spaces at the level of the Lusztig–Vogan modules.

Thank you for listening!