

The Diagrammatic Lusztig–Vogan Category for $SL(2, \mathbb{R})$

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Outline

1. Context and Motivation
2. Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$
3. Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$
4. Further work

Context and Motivation

Complex (simply-connected) Lie groups

Context and Motivation

Complex (simply-connected) Lie groups

BGG

Category $\mathcal{O}$

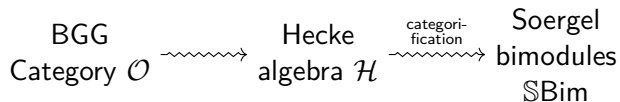
Context and Motivation

Complex (simply-connected) Lie groups

BGG
Category $\mathcal{O}$ $\rightsquigarrow$ Hecke
algebra $\mathcal{H}$

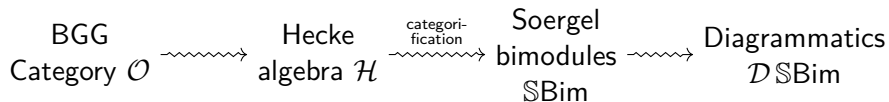
Context and Motivation

Complex (simply-connected) Lie groups



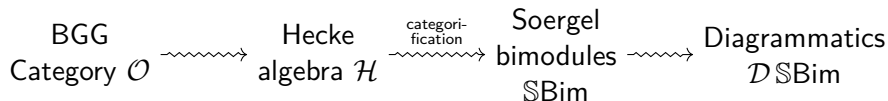
Context and Motivation

Complex (simply-connected) Lie groups



Context and Motivation

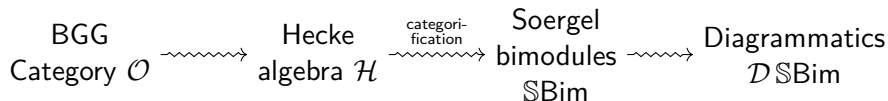
Complex (simply-connected) Lie groups



Real (reductive) Lie groups

Context and Motivation

Complex (simply-connected) Lie groups

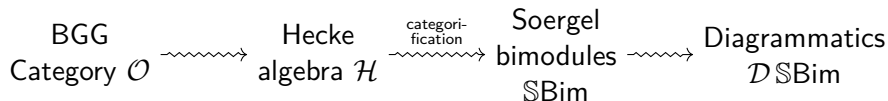


Real (reductive) Lie groups

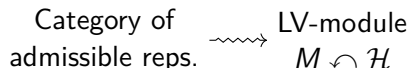
Category of
admissible reps.

Context and Motivation

Complex (simply-connected) Lie groups

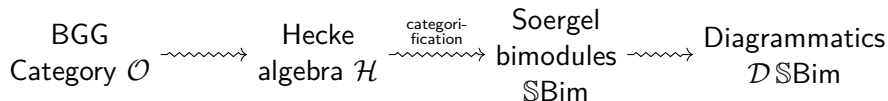


Real (reductive) Lie groups

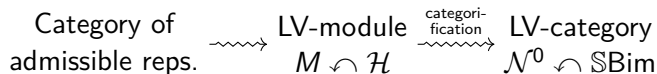


Context and Motivation

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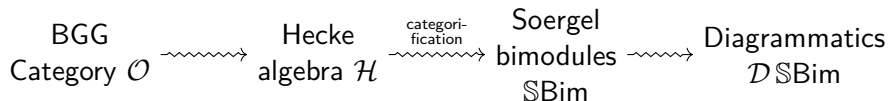


Real (reductive) Lie groups

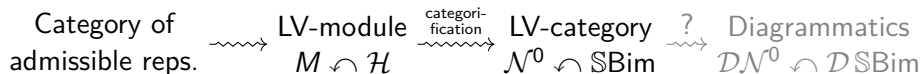


Context and Motivation

Complex (simply-connected) Lie groups



Real (reductive) Lie groups



Context and Motivation

Why diagrammatics?

- Diagrammatic methods enabled Elias & Williamson (2014) to give an algebraic proof of the Kazhdan–Lusztig conjecture (1979). This is more general than can be proved geometrically.
- Provided intuition for Williamson (2017) to discover counterexamples to Lusztig's conjecture (1980).
- $\mathcal{DSBim}$ can be studied in contexts where $\mathbb{S}Bim$ is not well-behaved, for example in fields of characteristic p .

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

The diagrammatic Hecke category $\mathcal{DH}$ (for $\mathfrak{sl}_2\mathbb{C}$) is a $\mathbb{Z}$ -linear monoidal category defined as follows.

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- *Objects*: generated by $\bullet$
i.e. $1, \bullet, \bullet\bullet := \bullet \otimes \bullet, \bullet\bullet\bullet, \bullet\bullet\bullet\bullet, \dots$

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- *Morphisms*: generated by



under relations...

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

Relations

$$\begin{array}{c}
 \begin{array}{c} | \\ \text{---} \bullet \end{array} = \begin{array}{c} | \\ | \end{array} = \begin{array}{c} \bullet \text{---} | \end{array}, \quad \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array}, \\
 \begin{array}{c} | \\ \bigcirc \end{array} = 0, \quad \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array},
 \end{array}$$

and arbitrary planar isotopy.

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$$\frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \\ \text{---} \bullet \text{---} \bullet \end{array}$$

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

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$$\boxed{\begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \end{array}}$$

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

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$$\frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \bullet \text{---} \bullet \\ | \\ \diagdown \quad \diagup \end{array} = \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \\ | \\ \diagup \quad \diagdown \end{array} + \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ | \\ \text{---} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \end{array}$$

$$\boxed{\begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array}}$$

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$$\frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \diagdown \quad \diagup \end{array} = \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ \bullet \text{---} \bullet \\ | \\ \diagup \quad \diagdown \end{array} + \frac{1}{2} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array}$$

$$\begin{array}{|c|} \bullet \\ \bullet \\ \hline \end{array} = 2 \begin{array}{|c|} \bullet \\ \bullet \\ \hline \end{array} - \begin{array}{|c|} \hline \bullet \\ \hline \end{array}$$

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$$\begin{aligned}
 \frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \\ \text{---} \bullet \text{---} \bullet \end{array} &= \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ \text{---} \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \\ \diagup \quad \diagdown \end{array} \\
 &= \begin{array}{c} | \quad | \\ \text{---} \bullet \quad \bullet \text{---} \end{array}
 \end{aligned}$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} | \\ \bullet \end{array}$$

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$$\frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \bullet \text{---} \bullet \\ | \\ \diagdown \quad \diagup \end{array} = \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \\ | \\ \diagup \quad \diagdown \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ | \\ \diagdown \quad \diagup \end{array}$$
$$= \begin{array}{c} | \quad | \\ \text{---} \bullet \quad \bullet \text{---} \end{array}$$

$\begin{array}{c} | \\ \text{---} \bullet \end{array} = \begin{array}{c} | \end{array} = \begin{array}{c} \bullet \text{---} \\ | \end{array}$

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$$\begin{aligned}
 \frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \end{array} &+ \frac{1}{2} \begin{array}{c} \text{---} \bullet \text{---} \bullet \\ | \\ \diagdown \quad \diagup \end{array} = \frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \bullet \text{---} \bullet \\ | \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \end{array} \\
 &= \begin{array}{c} | \quad | \\ \text{---} \bullet \quad \bullet \text{---} \\ | \quad | \end{array} = \begin{array}{c} | \quad | \end{array}
 \end{aligned}$$

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Theorem (Elias–Khovanov, 2010¹)

The Karoubi envelope of $\mathcal{DH}$ is equivalent to the category of Soergel Bimodules $\mathbb{S}\text{Bim}$ for $\mathfrak{sl}_2(\mathbb{C})$ as graded additive $\mathbb{R}$ -linear monoidal categories.

¹Ben Elias and Mikhail Khovanov. “Diagrammatics for Soergel categories”. In: *Int. J. Math. Math. Sci.* (2010), Art. ID 978635, 58.

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Definition

The diagrammatic LV-category $\mathcal{DN}^0$ (for $SL(2, \mathbb{R})$) is a $\mathbb{Z}$ -linear right module category over $\mathcal{DH}$ defined as follows. The right action by $\mathcal{DH}$ is right concatenation.

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- *Objects*: generated by $\mathbb{1}$ and $\circ$
i.e. $\mathbb{1}, \bullet, \bullet\bullet, \bullet\bullet\bullet, \bullet\bullet\bullet\bullet, \dots$ and $\circ, \circ\bullet, \circ\bullet\bullet, \dots$

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- *Morphisms*: generated by



under relations...

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Definition

Relations

The diagrammatic relations are as follows:

- Top-left relation: A vertical dotted red line with two red dots is equal to a vertical dotted red line with two red dots, followed by a comma.
- Top-right relation: A vertical solid red line with two red dots is equal to a vertical solid red line with two red dots minus a vertical solid red line with two red dots, followed by a comma.
- Bottom-left relation: A Y-junction with two dotted red lines on the left and one solid red line on the right is equal to a vertical dotted red line followed by a vertical solid red line, followed by a comma.
- Bottom-right relation: A vertical solid red line with two red dots is equal to zero, which is equal to a vertical dotted red line with two red dots, followed by a comma.

and arbitrary planar isotopy while dotted red lines never appear right of any red.

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2, \mathbb{R})$

Definition

Remark

This diagrammatic definition is not entirely new: Elias–Williamson² had constructed very similar diagrammatics for localisation of Soergel bimodules. The LV-category is a subcategory of this, with restrained objects, morphisms and relations.

²Ben Elias and Geordie Williamson. “Soergel calculus”. In: *Represent. Theory* 20 (2016), pp. 295–374.

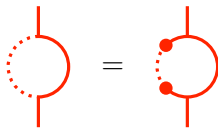
Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2, \mathbb{R})$

Example



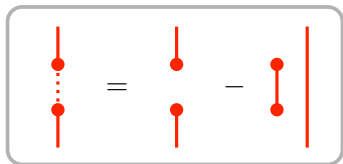
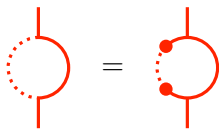
Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Example



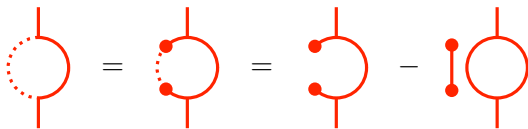
Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Example



Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

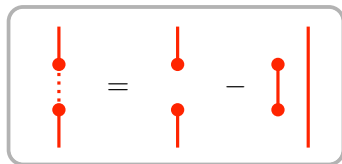
Example



A diagrammatic equation in the Lusztig–Vogan category for $SL(2, \mathbb{R})$. The equation is:

$$\text{Circle with dotted arc} = \text{Circle with two dots on dotted arc} = \text{Cup with two dots} - \text{Cap with two dots}$$

The first diagram is a circle with a vertical line passing through its center. A dotted arc is drawn on the left side of the circle. The second diagram is identical to the first, but the dotted arc is solid, and there are two red dots on it. The third diagram is a cup shape (the left half of a circle) with two red dots on its boundary. The fourth diagram is a cap shape (the right half of a circle) with two red dots on its boundary. The equation shows that the first two diagrams are equal, and the third diagram is equal to the fourth diagram with a minus sign.



A diagrammatic equation in the Lusztig–Vogan category for $SL(2, \mathbb{R})$, enclosed in a rounded rectangle. The equation is:

$$\text{Vertical line with two dots on dotted segment} = \text{Vertical line with two dots} - \text{Vertical line with two dots and a cap}$$

The first diagram is a vertical line with two red dots. The segment between the dots is dotted. The second diagram is a vertical line with two red dots. The third diagram is a vertical line with two red dots, and a cap shape (the right half of a circle) is attached to the right side of the line between the dots. The equation shows that the first diagram is equal to the second diagram minus the third diagram.

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Example

The diagram shows an equality between four terms, all drawn in red. Each term consists of a circle with a vertical line extending upwards and another extending downwards. The first term has a dotted arc on the left side of the circle. The second term has two solid dots on the left side of the circle. The third term has two solid dots on the left side of the circle, with the arc between them being solid. The fourth term is a minus sign followed by a circle with a vertical line on the left side connecting the two dots.

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} - \text{Diagram 4}$$

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Example

The diagram shows an equality between four terms. The first term is a circle with a vertical line extending upwards and another extending downwards; the left half of the circle is dashed. The second term is a solid circle with the same vertical lines, but with two red dots on its left side. The third term is a solid circle with the same vertical lines, with two red dots on its left side, but the circle is open on the left. The fourth term is a solid circle with the same vertical lines, with a vertical line segment passing through its center, also having two red dots on its left side. The equation is: $\text{dashed circle} = \text{dotted circle} = \text{open circle} - \text{crossed circle} = \text{vertical line}$.

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2, \mathbb{R})$

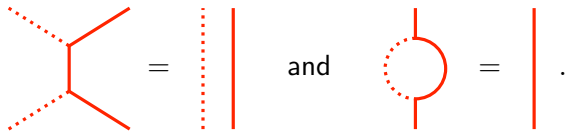
Theorem (Z. 2024)

The Karoubi envelope of $\mathcal{DN}^0$ is equivalent to the Lusztig–Vogan category $\mathcal{N}^0$ for $\mathrm{SL}(2, \mathbb{R})$ as graded additive $\mathbb{R}$ -linear right module categories over $\mathbb{S}\mathrm{Bim}$ (for $\mathfrak{sl}_2\mathbb{C}$).

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Proof idea

The backbone of the proof is the isomorphism $\bullet \simeq \circ \bullet$, given by the relations



The diagram shows two equations. The first equation shows a vertex with two solid red lines extending upwards and two dotted red lines extending downwards, equal to two parallel vertical lines, one solid red and one dotted red. The second equation shows a circle with a solid red line on the right and a dotted red line on the left, equal to a single solid red vertical line.

$$\begin{array}{c} \text{solid} \backslash \\ \text{dotted} / \end{array} = \begin{array}{c} \text{solid} \\ \text{dotted} \end{array} \quad \text{and} \quad \begin{array}{c} \text{solid} \\ \text{dotted} \end{array} = \text{solid}.$$

This reflects an isomorphism on the level of modules that allows us to construct a basis of morphisms from an existing one in $\mathcal{DH}$.

Further Work

- There is an isomorphism $SL(2, \mathbb{R}) \simeq SU(1, 1)$. A natural next step is to consider $SU(2, 1)$, with an aim to define diagrammatics for the infinite family $SU(n, 1)$.
- Understand the dimension of the general morphism spaces at the level of the Lusztig–Vogan modules.

Thank you for listening!