

Goals \mathbb{Q} : going between real and complex lie groups

- Real forms, involution
- Weyl group W, W_K

Real forms

• let $G(\mathbb{C})$ be complex reductive group.

largest smooth connected unipotent normal subgroup is trivial

(for any perfect field eg. \mathbb{C})
 \downarrow irr. poly has distinct roots in k

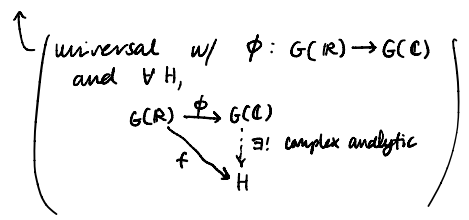
all finite dim representations are semisimple (so acts like a fin. group)

generalisation of semisimple (weaker cond.)
 "every smooth connected solvable normal subgroup is trivial"
 idea: reductive loosens this for nice theory: "semisimple up to abelian factor (centre)"

eg. $GL_n(\mathbb{C}), SL_n(\mathbb{C})$, not $B_n(\mathbb{C})$ upper triangular invertible

• We care about $G(\mathbb{R})$ such that complexification is $G(\mathbb{C})$
 (corresp to $\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C} = \mathfrak{g}_{\mathbb{C}}$)

There may be more than one!



• We will see these arise as $G(\mathbb{C})^{\sigma} = \{g \in G(\mathbb{C}) : g = \sigma(g)\}$ for an antiholomorphic involution σ

$\sigma: G(\mathbb{C}) \rightarrow G(\mathbb{C})$ homomorphism with $\sigma^2 = \text{id}$

st $\bar{\sigma}$ is holomorphic

analogy: holomorphic function that picks out the "real" stuff \uparrow complex differentiable

$\text{cf. } \mathbb{C}^{\text{Re}} = \mathbb{R}$

eg. $SL(2, \mathbb{C})$

anti holomorphic involutions

$\sigma_1: g \mapsto \bar{g}$

or $g \mapsto h^{-1} \overline{hgh^{-1}} h \mapsto h^{-1} \overline{h^{-1} \overline{hgh^{-1}} h} h^{-1} h = h^{-1} \overline{h^{-1} h} \overline{hgh^{-1}} \overline{h^{-1} h} h = g$

$\sigma_2: g \mapsto {}^t \bar{g}^{-1}$

- $G(\mathbb{C})$ acts on its automorphisms by composing with inner automorphisms
 $\text{int}(g): h \mapsto ghg^{-1}$
 where $g \cdot \phi = \text{int}(g) \circ \phi$
 $\phi \cdot g = \phi \circ \text{int}(g)$

but involutions are not always sent to involution

$$\text{eg } (\text{id} \circ \text{int}(g))^2 = \text{int}(g) \circ \text{int}(g): h \mapsto gghg^{-1}g^{-1} \neq h$$

If we look at conjugacy classes by this action, it works out

- Let σ be involution
- Conjugation

$$g \cdot \sigma \cdot g^{-1} = \text{int}(g) \circ \sigma \circ \text{int}(g^{-1})$$

is a convolution bc. $(\text{int}(g) \circ \sigma \circ \text{int}(g^{-1}))^2$

$$\begin{aligned} &= \text{int}(g) \circ \sigma \circ \text{int}(g^{-1}) \circ \text{int}(g) \circ \sigma \circ \text{int}(g^{-1}) \\ &= \text{int}(g) \circ \sigma \circ \sigma \circ \text{int}(g^{-1}) \\ &= \text{int}(g) \circ \sigma \circ \text{int}(g^{-1}) \end{aligned}$$

Turns out: this classifies anti-holomorphic involutions that give same real group

$$\left\{ \begin{aligned} \sigma(g) = \bar{g} &\Rightarrow \text{int}(h) \circ \sigma \circ \text{int}(h^{-1})(hg h^{-1}) = h \sigma(h^{-1}hg h^{-1})h^{-1} \\ &= h \sigma(g)h^{-1} \\ &= hgh^{-1} \\ \text{int}(h) \circ \sigma \circ \text{int}(h^{-1})(g) = \bar{g} &\Rightarrow h \sigma(h^{-1}gh)h^{-1} = \bar{g} \\ &\sigma(h^{-1}gh) = \overline{h^{-1}gh} \end{aligned} \right.$$

So $\text{int}(h): G(\mathbb{C})^\sigma \xrightarrow{\sim} G(\mathbb{C})^{\text{int}(h) \circ \sigma \circ \text{int}(h^{-1})}$

is isomorphism between these fixed groups.

- The above shows: well defined, and so is its inverse
- Clearly $\text{int}(h)$ and (the inverse) $\text{int}(h^{-1})$ are inverses so these groups are isomorphic
- "Same subgroup up to conjugation by $G(\mathbb{C})$ "

↑ via regular multiplication

eg. $SL_2(\mathbb{C})$

$$\begin{aligned} \text{had } \sigma: g &\mapsto \bar{g} & \xrightarrow{\text{int}(h)} & g &\mapsto h \overline{h^{-1}gh} h^{-1} \\ & & & &\mapsto h \overline{h^{-1}gh} h^{-1} \\ & & & &= h \overline{h^{-1}gh} h^{-1} \\ & & & &= h h^{-1} g h h^{-1} \\ & & & &= g \end{aligned}$$

What is $SL_2(\mathbb{C})^\sigma$?

- σ applies to all entries so this is $SL_2(\mathbb{R}) \leftarrow$ real Lie group!

What about $SL_2(\mathbb{C})^{\sigma_c}$?

where $\sigma_c: g \mapsto {}^t \bar{g}^{-1}$

Note: these are the only ones

• det 1 complex matrices M where

$${}^t \bar{M}^{-1} = M$$

$$\Leftrightarrow {}^t \bar{M} = M^{-1}$$

unitary matrices!

$$SL_2(\mathbb{C})^{\sigma_c} = SU(2)$$

↑ complex matrices but real lie group!
 (something about defining functions being smooth but non-holomorphic)

• Turns out that

$$\left\{ \begin{array}{l} \text{antiholomorphic involutions} \\ \text{of } G(\mathbb{C}) \end{array} \right\} / G(\mathbb{C}) \leftrightarrow \left\{ \begin{array}{l} \text{holomorphic involutions} \\ \text{of } G(\mathbb{C}) \end{array} \right\} / G(\mathbb{C})$$

↑ "conjugation by $\text{Int}(g)$ "-action

FACT f, g involutions
 Then $f \circ g$ involution iff $f \circ g = g \circ f$

$$\sigma \xrightarrow{\text{choice of } \sigma_c \text{ s.t. } \sigma \circ \sigma_c = \theta} \sigma \circ \sigma_c := \theta$$

$$\theta \circ \sigma_c \xleftarrow{\text{choice of } \theta \text{ s.t. } \theta \circ \sigma_c = \sigma} \theta$$

↑ "Cartan involution" of σ

THM (Knaupp, 7HM6.16) for Lie Algebras
 Let θ be Cartan involution, σ any involution.
 There exists $\phi \in \text{Int}(\sigma)$ s.t. $\phi \theta \phi^{-1}$ commutes with σ .

• Probably lifts to lie groups

THM (Cartan)

There exists a unique involution $\sigma_{\mathbb{C}}$ of $G(\mathbb{C})$ such that σ_c is antiholomorphic and $G(\mathbb{C})^{\sigma_{\mathbb{C}}}$ is compact.
 This defines the above bijection. (in $G(\mathbb{C})$)

In our case, the map σ_c is $\sigma_c: g \mapsto {}^t \bar{g}^{-1}$.

eg $SL_2(\mathbb{C})$

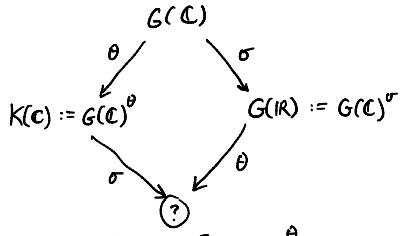
$$\begin{array}{ccc} \text{antiholomorphic} & & \text{holomorphic} \\ \sigma_1: g \mapsto \bar{g} & \xleftrightarrow{\sigma_c} & \theta_1 = \sigma \circ \sigma_c: g \mapsto {}^t \bar{g}^{-1} \\ \sigma_2: g \mapsto {}^t \bar{g}^{-1} & \xleftrightarrow{\sigma_1} & \theta_2: g \mapsto g \end{array}$$

• We know σ -fixed subgroup = real group
 What do these Cartan involutions do?

FACT • $K := G(\mathbb{C})^\theta$ is a closed subgroup of $G(\mathbb{C})$

We also know $\sigma \circ \theta = \sigma \circ \sigma \circ \sigma$
 $= \sigma_c$
 $= \sigma_c \circ \sigma \circ \sigma$
 $= \theta \circ \sigma$

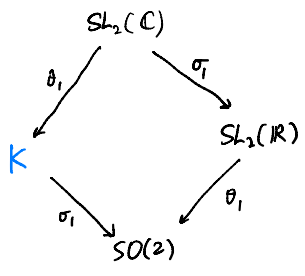
so we could draw (arrows are not maps)



$K(\mathbb{R}) := K(\mathbb{C})^\sigma = G(\mathbb{R})^\theta$

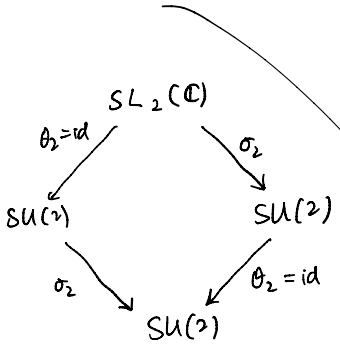
FACT $K(\mathbb{R})$ is maximal compact subgroup of $G(\mathbb{R})$
 whose complexification is $K(\mathbb{C})$

Example



- $SL_2(\mathbb{C})^\theta$:
 ${}^t M^{-1} = M \iff M^{-1} = M^T$
 \iff for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
 $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$
 $\iff \begin{cases} a=d \\ b=-c \end{cases}$
 $\iff \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{C}, a^2 + b^2 = 1 \right\} = K$

- $SL_2(\mathbb{R})^\theta$: $M^{-1} = M^T$
 orthogonal matrices
 with determinant 1
 $\Rightarrow SO(2)$
- $K^{\sigma_1} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1, \overline{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, a, b \in \mathbb{C} \right\}$
 $= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1, a, b \in \mathbb{R} \right\}$
 $= SO(2)$



Then $W = N_G(T)/Z_G(T)$ Weyl group for G , T torus in G

$W_K = N_K(T_K)/Z_K(T_K)$ Weyl group for K , T_K torus in K

where $N_G(T) = \{g \in G : gtg^{-1} \in T, \forall t \in T\}$ Normaliser

$T = Z_G(T) = \{g \in G : gtg^{-1} = t, \forall t \in T\}$ Centraliser

LEM 6.1.2 " T is θ -stable" $\iff \theta(t) \in T, \forall t \in T$

So θ acts on $W = N(T)/T$

- well def: if $g \in N(T), gtg^{-1} \in T \Rightarrow \theta(g)t\theta(g^{-1}) \stackrel{\text{group hom}}{\downarrow} \theta(gtg^{-1}) \stackrel{T \text{ is } \theta\text{-stable}}{\downarrow} \in T, \forall t \in T$
- an involution, since α is involution on G

$W^\theta = \theta$ -fixed points of W .