Goals Q: going between real and complex Lie groups

- Real forms, involution
- Weyl group W, W_K

Real forms

- Let $G(\mathbb{C})$ be complex reductive group.

$\updownarrow$

largest smooth connected unipotent normal subgroup is trivial } generalization of semisimple (weaker cond.) "every smooth connected solvable normal subgroup is trivial"

idea: reductive loosens this for nicer theory: "semisimple up to abelian factor (centre)"

$\updownarrow$

(for any perfect field eg. $\mathbb{C}$)

$\forall$ irred. poly has distinct roots in $\bar{k}$

all finite dim representations are semisimple (so acts like a fin. group)

eg $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, not $B_n(\mathbb{C})$ upper triangular invertible

- We care about $G(\mathbb{R})$ such that complexification is $G(\mathbb{C})$
(corresp to $\mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C} = \mathfrak{g}_{\mathbb{C}}$)

universal w/ $\phi : G(\mathbb{R}) \to G(\mathbb{C})$ and $\forall H$,

$$\begin{tikzcd} G(\mathbb{R}) \arrow[r, "\phi"] \arrow[dr, "f"'] & G(\mathbb{C}) \arrow[d, dashed, "\exists!\ \text{complex analytic}"] \\ & H \end{tikzcd}$$

There may be more than one!

- We will see these arise as $G(\mathbb{C})^{\sigma} = \{ g \in G(\mathbb{R}) : g = \sigma(g) \}$ for an antiholomorphic involution σ

$\sigma : G(\mathbb{C}) \to G(\mathbb{C})$
s.t. $\bar{\sigma}$ is holomorphic

homomorphism with $\sigma^2 = \mathrm{id}$

complex differentiable

analogy: holomorphic function that picks out the "real" stuff

$\boxed{\text{cf. } \mathbb{C}^{(x \mapsto \bar{x})} = \mathbb{R}}$

eg. $SL(2, \mathbb{C})$

anti holomorphic involutions

$\sigma_1 : g \longmapsto \bar{g}$ or $g \longmapsto h^{-1}\overline{hgh^{-1}}h \longmapsto h^{-1}\overline{h(h^{-1}\overline{hgh^{-1}}h)h^{-1}}h$
$= h^{-1}\overline{hh^{-1}}hgh^{-1}\overline{hh^{-1}}h$
$= g$

$\sigma_2 : g \longmapsto {}^{t}\bar{g}^{-1}$

- $G(\mathbb{C})$ acts (both sides) on it's automorphisms by composing with innerautomorphisms $\mathrm{int}(g): h \longmapsto ghg^{-1}$

where $g \cdot \phi = \mathrm{int}(g) \circ \phi$
$\phi \cdot g = \phi \circ \mathrm{int}(g)$

↑ in particular, involutions

but involutions are not always sent to involutions

eg $(id \circ \mathrm{int}(g))^2 = \mathrm{int}(g) \circ \mathrm{int}(g) : h \longmapsto gghg^{-1}g^{-1} \neq h$

If we look at conjugacy classes by this action, it works out

- let σ be involution
- Conjugation

$$g \cdot \sigma \cdot g^{-1} = \mathrm{int}(g) \circ \sigma \circ \mathrm{int}(g^{-1})$$

is a convolution bc. $\left(\mathrm{int}(g) \circ \sigma \circ \mathrm{int}(g^{-1})\right)^2$

$$= \mathrm{int}(g) \circ \sigma \circ \cancel{\mathrm{int}(g^{-1})} \circ \cancel{\mathrm{int}(g)} \circ \sigma \circ \mathrm{int}(g^{-1})$$
$$= \mathrm{int}(g) \circ \sigma \circ \sigma \circ \mathrm{int}(g^{-1})$$
$$= \mathrm{int}(g) \circ \sigma \circ \mathrm{int}(g^{-1})$$

Turns out: this classifies antiholomorphic involutions that give same real group

$\sigma(g) = g \Rightarrow \mathrm{int}(h) \circ \sigma \circ \mathrm{int}(h^{-1})(hgh^{-1}) = h\sigma(h^{-1}hgh^{-1}h)h^{-1}$
$= h\sigma(g)h^{-1}$
$= hgh^{-1}$

$\mathrm{int}(h) \circ \sigma \circ \mathrm{int}(h^{-1})(g) = g \Rightarrow h\sigma(h^{-1}gh)h^{-1} = g$
$\sigma(h^{-1}gh) = h^{-1}gh$

So $\mathrm{int}(h): G(\mathbb{C})^{\sigma} \xrightarrow{\sim} G(\mathbb{C})^{\mathrm{int}(h) \circ \sigma \circ \mathrm{int}(h^{-1})}$

is isomorphism between these fixed groups.

- The above shows: well defined, and so is it's inverse
- Clearly $\mathrm{int}(h)$ and (the inverse) $\mathrm{int}(h^{-1})$ are inverses so these groups are isomorphic
- "same subgroup up to conjugation by $G(\mathbb{C})$"

↑ via regular multiplication

eg. $SL_2(\mathbb{C})$

had $\sigma_1 : g \longmapsto \bar{g}$ $\overset{\mathrm{int}(h)}{\rightsquigarrow}$ $g \longmapsto h\overline{h^{-1}gh}h^{-1}$

$\longmapsto h\overline{\cancel{h^{-1}}\cancel{h}\overline{h^{-1}gh}\cancel{h^{-1}}\cancel{h}}h^{-1}$ ← involution!

$= h\overline{\overline{h^{-1}gh}}h^{-1}$
$= hh^{-1}ghh^{-1}$
$= g$

What is $SL_2(\mathbb{C})^{\sigma_1}$?

- σ applies to all entries so this is $SL_2(\mathbb{R})$ ← real lie group!

What about $SL_2(\mathbb{C})^{\sigma_2}$?

where $\sigma_2 : g \longmapsto {}^t\overline{g}^{-1}$

Note: these are the only ones

- det 1 complex matrices M where

$$ {}^t\overline{M}^{-1} = M $$

$$ \Leftrightarrow {}^t\overline{M} = M^{-1} $$

unitary matrices!

$$ SL_2(\mathbb{C})^{\sigma_2} = SU(2) $$

complex matrices but real lie group!

(something about defining functions being smooth but non-holomorphic)

- Turns out that

$$ \left\{ \begin{array}{c} \text{antiholomorphic involutions} \\ \text{of } G(\mathbb{C}) \end{array} \right\} \Big/ G(\mathbb{C}) \longleftrightarrow \left\{ \begin{array}{c} \text{holomorphic involutions} \\ \text{of } G(\mathbb{C}) \end{array} \right\} \Big/ G(\mathbb{C}) $$

"conjugation by Int($\mathfrak{g}$)"-action

$$ \sigma \xmapsto{\text{choice of } \sigma \text{ so } \sigma\circ\sigma_c = \sigma_c\circ\sigma} \sigma\circ\sigma_c := \theta $$

"Cartan Involution" of σ

$$ \theta\circ\sigma_c \xleftarrow{\text{choice of } \theta \text{ st. } \theta\circ\sigma_c = \sigma_c\circ\theta} \theta $$

FACT f, g involutions

Then $f\circ g$ involution iff $f\circ g = g\circ f$

THM (Knapp, THM 6.16) for Lie Algebras

Let θ be Cartan Involution, σ any involution.
There exists $\phi \in \mathrm{Int}(\mathfrak{g})$ s.t. $\phi\theta\phi^{-1}$ commutes with σ.

- Probably lifts to Lie groups

THM (Cartan)

There exists a unique involution σ_c of $G(\mathbb{C})$ such that σ_c is antiholomorphic and $G(\mathbb{C})^{\sigma_c}$ is compact (in $G(\mathbb{C})$).
This defines the above bijection.

In our case, the map σ_c is $\sigma_2 : g \longmapsto {}^t\overline{g}^{-1}$.

eg $SL_2(\mathbb{C})$

antiholomorphic		holomorphic
$\sigma_1 : g \longmapsto \overline{g}$	$\xleftrightarrow{\sigma_2}$	$\theta_1 = \sigma\circ\sigma_c : g \longmapsto {}^t g^{-1}$
$\sigma_2 : g \longmapsto {}^t\overline{g}^{-1}$	$\xleftrightarrow{\sigma_2}$	$\theta_2 : g \longmapsto g$

- We know σ-fixed subgroup = real group
What do these Cartan involutions do?

FACT • $K := G(\mathbb{C})^\theta$ is a closed subgroup of $G(\mathbb{C})$

We also know $\sigma \circ \theta = \sigma \circ \sigma \circ \sigma \circ \sigma_c$
$= \sigma_c$
$= \sigma_c \circ \sigma \circ \sigma$
$= \theta \circ \sigma$

so we could draw (arrows are not maps)

$$\begin{tikzcd} & G(\mathbb{C}) \arrow[dl, "\theta"'] \arrow[dr, "\sigma"] & \\ K(\mathbb{C}) := G(\mathbb{C})^\theta \arrow[dr, "\sigma"'] & & G(\mathbb{R}) := G(\mathbb{C})^\sigma \arrow[dl, "\theta"] \\ & ? & \end{tikzcd}$$

$K(\mathbb{R}) := K(\mathbb{C})^\sigma = G(\mathbb{R})^\theta$

FACT $K(\mathbb{R})$ is maximal compact subgroup of $G(\mathbb{R})$ whose complexification is $K(\mathbb{C})$

Example

$$\begin{tikzcd} & SL_2(\mathbb{C}) \arrow[dl, "\theta_1"'] \arrow[dr, "\sigma_1"] & \\ K \arrow[dr, "\sigma_1"'] & & SL_2(\mathbb{R}) \arrow[dl, "\theta_1"] \\ & SO(2) & \end{tikzcd}$$

- $SL_2(\mathbb{C})^\theta$:

$^tM^{-1} = M \iff M^{-1} = M^T$

$\iff$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\iff \begin{cases} a = d \\ b = -c \end{cases}$

$\iff \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{C},\ a^2 + b^2 = 1 \right\} = K$

- $SL_2(\mathbb{R})^\theta$: $M^{-1} = M^T$
orthogonal matrices with determinant 1
$\Rightarrow SO(2)$

- $K^{\sigma_1} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1,\ \overline{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}} = \begin{pmatrix} a & b \\ b & a \end{pmatrix},\ a, b \in \mathbb{C} \right\}$

$= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1,\ a, b \in \mathbb{R} \right\}$

$= SO(2)$

$$\begin{tikzcd} & SL_2(\mathbb{C}) \arrow[dl, "\theta_2 = id"'] \arrow[dr, "\sigma_2"] & \\ SU(2) \arrow[dr, "\sigma_2"'] & & SU(2) \arrow[dl, "\theta_2 = id"] \\ & SU(2) & \end{tikzcd}$$

Then $W = N_G(T)/Z_G(T)$ Weyl group for G, T torus in G

$W_K = N_K(T_K)/Z_K(T_K)$ Weyl group for K, T_K Torus in K

where $N_G(T) = \{ g \in G : gtg^{-1} \in T, \ \forall t \in T \}$ Normaliser

$T = Z_G(T) = \{ g \in G : gtg^{-1} = t, \ \forall t \in T \}$ Centraliser

<u>LEM 6.1.2</u> "T is θ-stable" $\Longleftrightarrow$ $\theta(t) \in T, \ \forall t \in T$

So θ acts on $W = N(T)/T$

- well def : if $g \in N(T)$, $gtg^{-1} \in T \Rightarrow \theta(g)\, t\, \theta(g^{-1}) \overset{\text{group hom}}{=} \theta(gtg^{-1}) \overset{T \text{ is } \theta\text{-stable}}{\in} T$, $\forall t \in T$
- an involution, since θ is involution on G

W^θ = θ-fixed points of W.