Diagrammatic Lusztig–Vogan Categories

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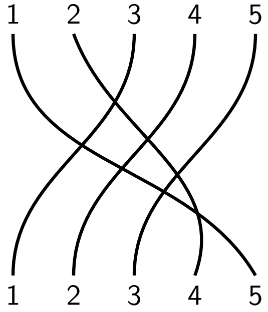
Outline

1. String diagrams in mathematics
2. Representation theory motivation
3. Diagrammatic Lusztig–Vogan categories for $\mathrm{SU}(n, 1)$

String diagrams in mathematics

1. Permutations

$$S_5 \ni \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} = (135)(24)$$



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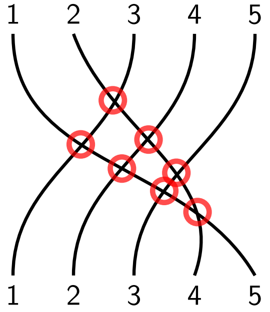


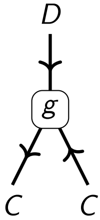
$$\operatorname{sign}\big((135)(24)\big) = (-1)^7 = -1$$

String diagrams in mathematics

2. **Monoidal categories** (rigid, pivotal)

$$\begin{tikzcd} D \\ C \arrow[u, "f"] \end{tikzcd}$$

$$\begin{tikzcd} D \\ C \otimes C^{*} \arrow[u, "g"] \end{tikzcd}$$



String diagrams in mathematics

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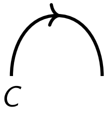
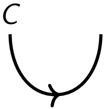
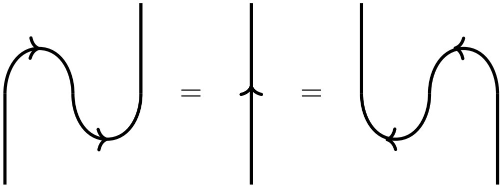


$$C^* \otimes C \longrightarrow \mathbb{1}$$

,

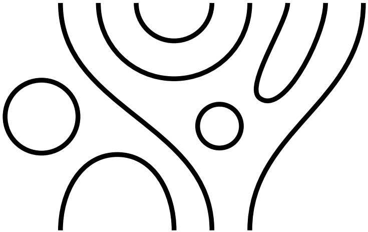


$$\mathbb{1} \longrightarrow C \otimes C^*$$



String diagrams in mathematics

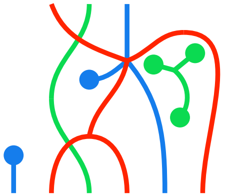
2. Monoidal categories (rigid, pivotal)
e.g. the Temperley–Lieb–Jones category



- Links to $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))$ and produces a knot invariant.

String diagrams in mathematics

3. **Soergel bimodules** (an additive monoidal category)

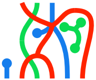


- Algebraic proof that generalises the Kazhdan–Lusztig conjecture (Elias–Williamson)
- Counterexample to Lusztig's conjecture (Williamson)

Representation theory motivation

Representation theory motivation

Diagrammatic Soergel bimodules $\mathcal{DH}$ $\rightsquigarrow$ Reps of complex semisimple Lie algebras



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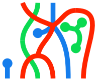

Admissible reps of real reductive Lie groups

Representation theory motivation

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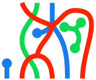

Diagrammatic Lusztig–Vogan categories $\mathcal{D}\tilde{\mathcal{N}}^0$ $\overset{???}{\rightsquigarrow}$ Admissible reps of real reductive Lie groups

???

Two colour diagrammatic Soergel bimodules

First, we need the diagrammatic Soergel bimodules $\mathcal{DH}$ for

$$S_3 = \langle {\color{red}\bullet}, {\color{blue}\bullet} \mid {\color{red}\bullet}^2 = {\color{blue}\bullet}^2 = 1, {\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet} = {\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet} \rangle.$$

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$\mathcal{DH}$ is a categorical version: ($\mathbb{C}$-linear monoidal)

- *Objects:* $\mathbb{1}, {\color{red}\bullet}, {\color{blue}\bullet}, {\color{red}\bullet}{\color{red}\bullet}, {\color{red}\bullet}{\color{blue}\bullet}, {\color{blue}\bullet}{\color{red}\bullet}, {\color{blue}\bullet}{\color{blue}\bullet}, {\color{red}\bullet}{\color{red}\bullet}{\color{red}\bullet}, {\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet}, \ldots$

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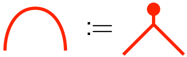
These combine into



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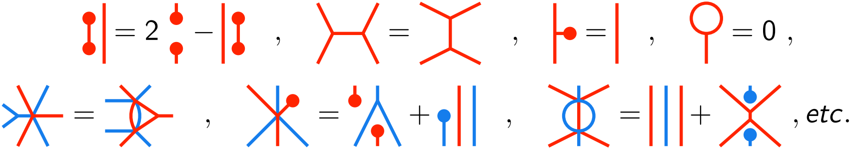
- *Morphisms:* generated by

- *Relations:* Planar isotopy,



Two colour diagrammatic Lusztig–Vogan category

We take the subgroup

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- *Objects:* generated by minimal coset representatives

$$\mathbb{1}, {\color{blue}\circ}, {\color{blue}\circ}{\color{red}\circ} \qquad\qquad \curvearrowleft \mathbb{1}, {\color{red}\bullet}, {\color{blue}\bullet}, {\color{red}\bullet}{\color{blue}\bullet}, \ldots$$

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The wall corresponds to the quotient:

$${\color{red}\bullet} = 1$$

$${\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet} = {\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet} = {\color{blue}\bullet}{\color{red}\bullet}$$

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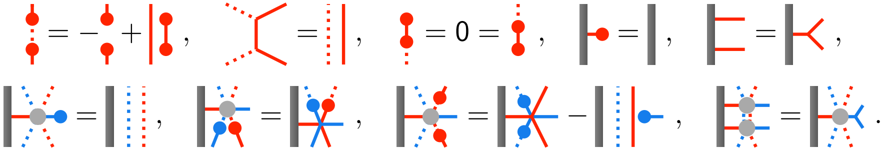
- *Morphisms:* generated by

- *Relations:* Limited planar isotopy,



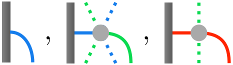
n-colour diagrammatic Lusztig–Vogan categories

We can generalise to

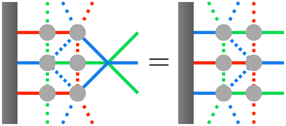
$$\langle s_1, \ldots, s_{n-1} \rangle < \langle s_1, \ldots, s_n \mid s_i^2 = 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \rangle$$
$$= S_{n+1}.$$

e.g. if $n = 3$, $\langle$•, • $\mid \ldots \rangle < \langle$•, •, • $\mid \ldots \rangle = S_4$

- *Objects:* generated by $\mathbb{1}$, ○, ○○, ○○○
- *New Morphisms:*



- *New Relation:* with 3-colors



Back to representation theory

Theorem (Z.)

There is an equivalence of $\mathbb{C}$-linear module categories $\mathcal{D}\tilde{\mathcal{N}}^0 \simeq \tilde{\mathcal{N}}^0$ for the real groups $\mathrm{SU}(n,1)$ and $n \geq 1$.

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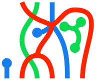
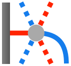
Diagrammatic Soergel bimodules $\mathcal{DH}$ $\rightsquigarrow$ Reps of complex semisimple Lie algebras



Diagrammatic Lusztig–Vogan categories $\mathcal{D}\tilde{\mathcal{N}}^0$ $\rightsquigarrow$ Admissible reps of real reductive Lie groups



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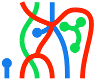

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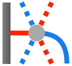

Thank you for your attention.