Lecture 3: Drawing monoidal categories

Outline:
- diagrams for monoidal categories
- duals
- Trace, braided and symmetric monoidal categories
- Monoid objects and Frobenius objects.

Recall the definition of a monoidal category:
- category $\mathcal{C}$
- bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
- natural isomorphisms

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \qquad \text{associator}$$

$$l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X \qquad \text{left unitor}$$

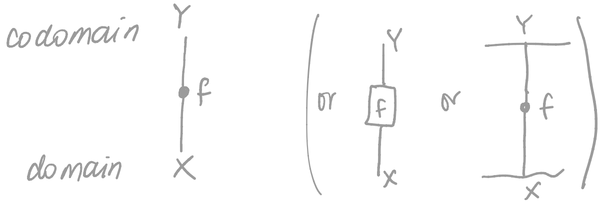
$$r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X \qquad \text{right unitor}$$

satisfying ⬠ and ▽

"Diagrams" are graphical representations of morphisms. Why care?

Diagrams for morphisms

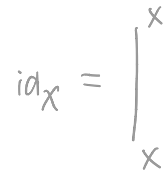
- $f : X \to Y$



- composition

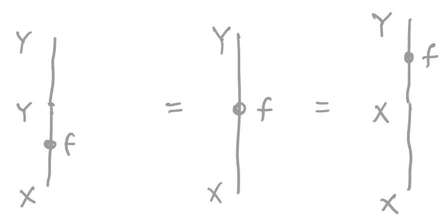
eg. $f : X \to Y$, $g : Y \to Z$

- identity of composition



sensible because

$$id_Y \circ f = f = f \circ id_X$$



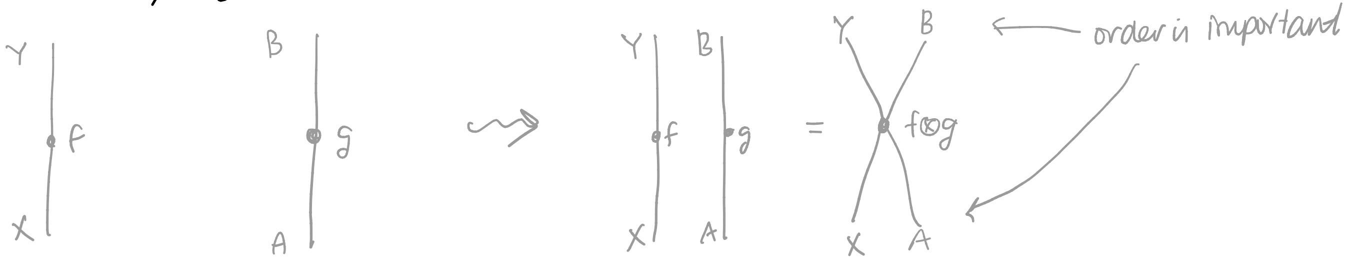
commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X"'] \arrow[dr, "f"] & Y \arrow[d, "id_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

we can think of objects X as ... the identity $id_X =$ 

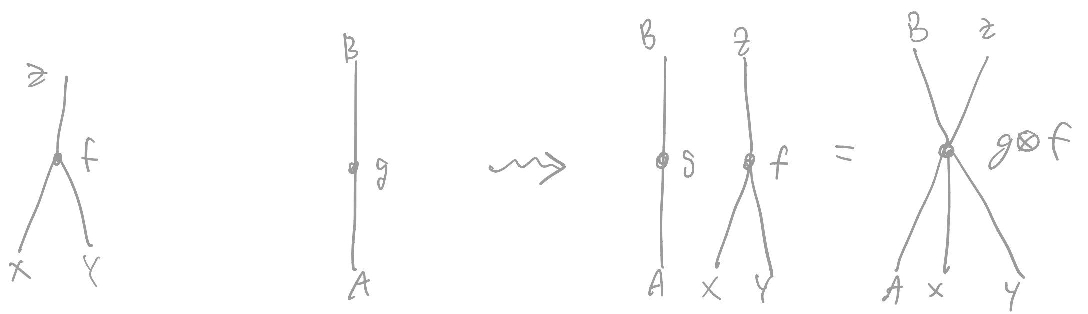
- Tensor product, the feature of monoidal categories

eg $f: X \to Y$, $g: A \to B$ then $f \otimes g: X \otimes A \to Y \otimes B$

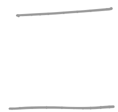


(we restrict to strict monoidal categories)

eg. $f: X \otimes Y \to Z$, $g: A \to B$ then $g \otimes f: A \otimes X \otimes Y \to B \otimes Z$

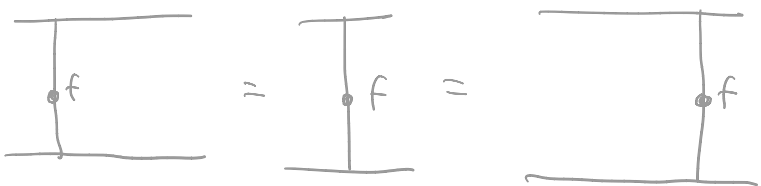


- Identity of tensor product

$\mathrm{id}_{\mathbb{1}} =$ 

sensible because (in a strict monoidal cat.)

$$f \otimes \mathrm{id}_{\mathbb{1}} = f = \mathrm{id}_{\mathbb{1}} \otimes f$$



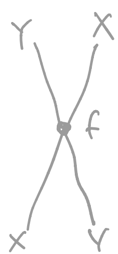
(commutative diagram (first equality)

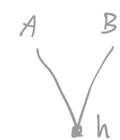
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal, "r_X"'] & Y \arrow[d, equal, "r_Y"] \\ X \otimes \mathbb{1} \arrow[r, "f \otimes \mathrm{id}_{\mathbb{1}}"'] & Y \otimes \mathbb{1} \end{tikzcd}$$

by naturality of $r : - \otimes \mathbb{1} \to \mathrm{Id}$)

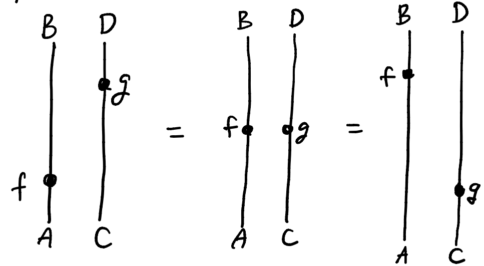
- More examples

$f : X \otimes \mathbb{1} \otimes Y \longrightarrow \mathbb{1} \otimes Y \otimes X$, $g : X \to \mathbb{1}$, $h : \mathbb{1} \to A \otimes B$





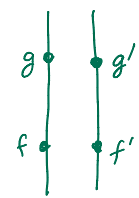
- Bifunctoriality implies, for $f: A \to B$, $g: C \to D$,

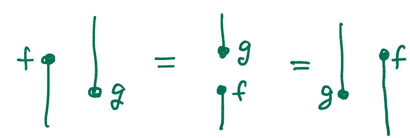


"Interchange law"

- <u>Exercise</u> Draw the corresponding commutative diagram to check this

<u>Exercise</u> Show that the two interpretations of the following agree



<u>Exercise</u> Show the following relation 

- We have "Rectilinear isotopy"

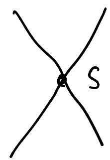
Example 1

The *symmetric category* Sym is a (strict symmetric) monoidal category

objects generated by tensor products of $\bullet$.

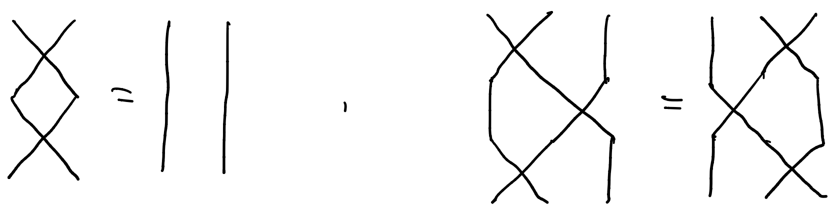
The morphisms are generated by

using horizontal and vertical concatenation and identity morphism

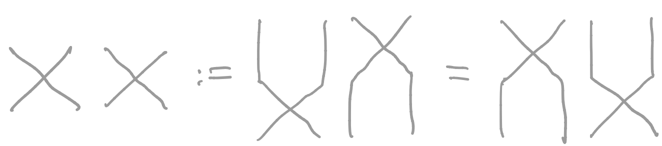


under the local relations

can be applied to subdiagrams



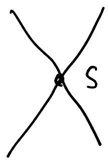
and with rectilinear isotopy, ie. diagrams similar to



Example 1 (cont.)

Notice: all morphisms in Sym are endomorphisms

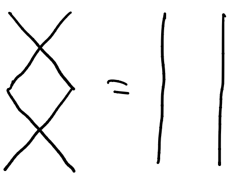
In fact we have $\mathrm{End}(1^{\otimes n}) \simeq S_n$ ...



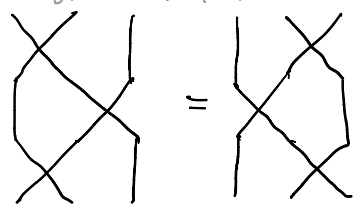
represents transposition $s \in S_2$

doesn't exist in S_n, we define it to be one of these

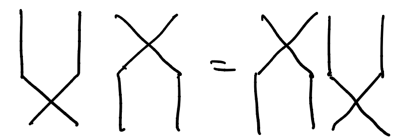
"braid relation"



$s^2 = 1$



$s_1 s_2 s_1 = s_2 s_1 s_2$

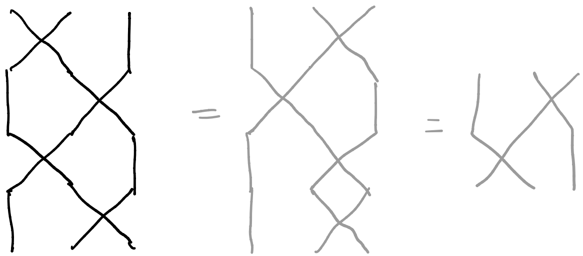


$s_1 s_3 = s_3 s_1$

This mirrors the coxeter presentation of S_n (the other generators are tensors of s with identity and other relations follow by locally applying these, and rectilinear isotopy)

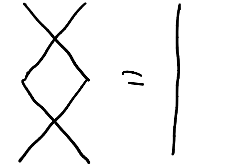
Example 1 (cont.)

Let's simplify the following morphism



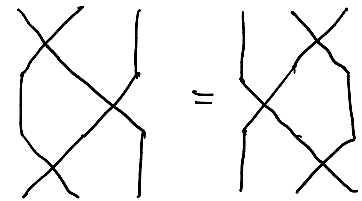
$$(12)(23)(12)(23) = (23)(12)(23)(23) = (23)(12)$$

we can also do this by looking where the strands connect

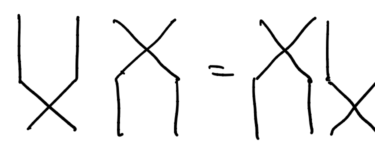


$$s^2 = 1$$

"braid relation"



$$s_1 s_2 s_1 = s_2 s_1 s_2$$

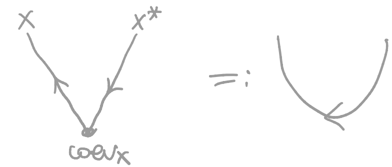


$$s_1 s_3 = s_3 s_1$$

<u>Duals</u>

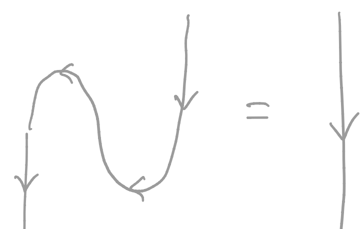
- Let X^* be left dual of X. Recall this is defined by morphisms

$$ev_X : X^* \otimes X \to \mathbb{1} \qquad coev_X : \mathbb{1} \to X \otimes X^*$$



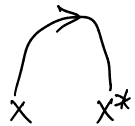
such that these triangles commute

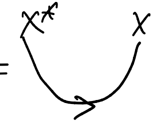
$$\begin{tikzcd}
X \arrow[r, "coev_X \otimes id_X"] \arrow[dr, equal] & X \otimes X^* \otimes X \arrow[d, "id_X \otimes ev_X"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
X^* \arrow[r, "id_{X^*} \otimes coev_X"] \arrow[dr, equal] & X^* \otimes X \otimes X^* \arrow[d, "ev_X \otimes id_{X^*}"] \\
& X^*
\end{tikzcd}$$



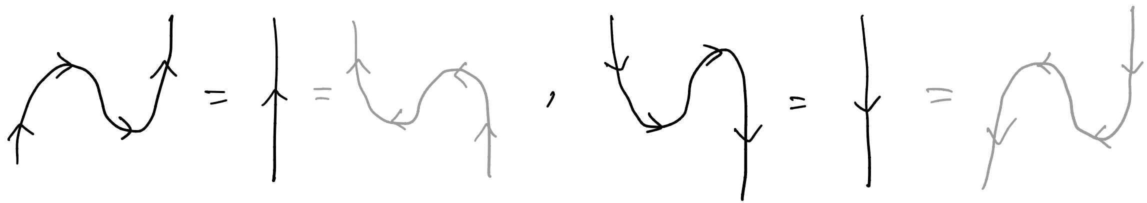
"zig zag relations"

• if X is <u>also</u> a left dual of X^* (ie X, X^* are biadjoint) we have

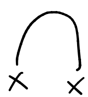
$ev =$ 

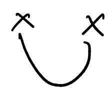
$coev =$ 

and

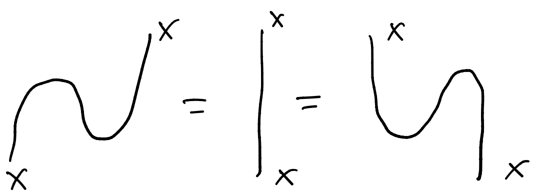


• if X is self dual, $\uparrow = \downarrow$ so we can omit the arrows. Then

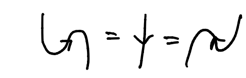
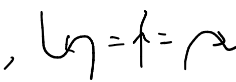
$ev_X =$ 

, $coev_X =$ 

and

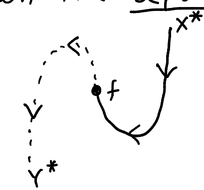


Isotopy

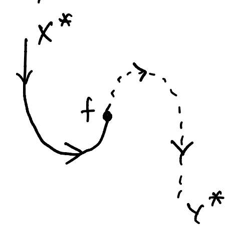
- We saw a glimpse of equivalence under true isotopy with  , 
- With more conditions we can view diagrams under general isotopy...

Suppose X, X^* are biadjoint and likewise with Y, Y^*
$(\uparrow, \downarrow)$ $(\uparrow, \downarrow)$

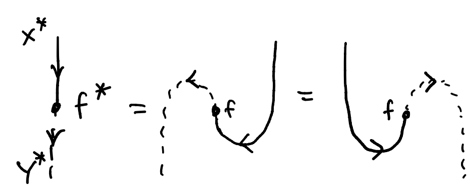
DEF Let $f: X \to Y$ be a morphism. Then the <u>left dual</u> of f is $f^*: Y^* \to X^*$ defined to be



and the <u>right dual</u> of f is ${}^*f: Y^* \to X^*$ defined to be



If $f^* = {}^*f$, we say f is <u>cyclic</u> and draw



• In general, morphisms need not be cyclic

PROP (Intro to Soergel Bimodules Prop 7.18)

In a monoidal category where all objects are biadjoints and all morphisms are cyclic, diagrams up to isotopy unambiguously represent a morphism.

This is what we wanted!

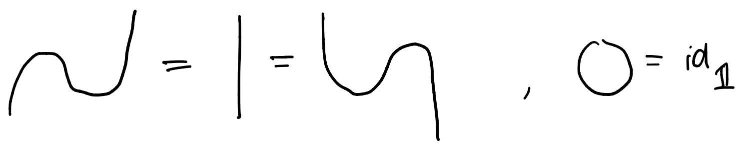
Remark: We could say that "rigid" in tensor category definition is so that we get closer to this desirable property.

Example 2

The Temperley-Lieb category TL_1 is a (strict) monoidal category with objects generated by $\bullet$, morphisms generated by

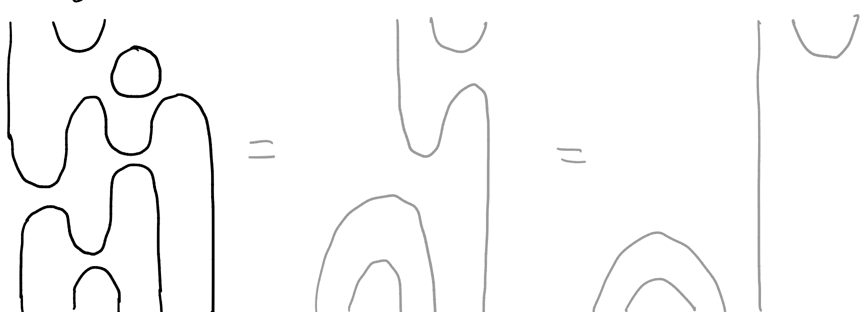
$$\cap : \bullet \otimes \bullet \longrightarrow \mathbb{1} \quad , \quad \cup : \mathbb{1} \longrightarrow \bullet \otimes \bullet$$

with relations



<u>Note</u> this implies that $\bullet$ is self dual

Let's simplify this morphism



Example 2 (cont.)

Exercise Show that TL_1 is rigid and pivotal. Draw their defining morphisms and relations.

- In fact every object is self dual, in particular biadjoint, and all morphisms are cyclic, So we have general isotopy on TL_1!

  Ex. convince yourself of this

- Remark We can construct TL_{-2} by making this category $\mathbb{C}$-linear and setting $\bigcirc = -2\,\mathrm{id}_{\mathbb{1}}$. Then TL_{-2} is equivalent to the category with objects $V^{\otimes n}$ (where $V=\mathbb{C}^2$ is the standard representation of $SL(2,\mathbb{C})$) and morphisms being all $SL(2,\mathbb{C})$-equivariant morphisms between them.

An aside on 2-category diagrams

- The typical example is Cat, category of (small) categories

strand = functor

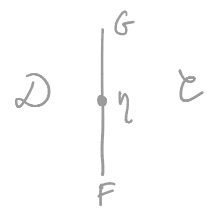
dots = natural transformation

vertical stacking = vertical composition

horizontal concatenation = horizontal composition

and regions = categories (objects in Cat)

- eg. natural transformation $\eta : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is drawn



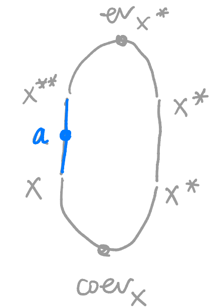
- All the things we did before also extends to this context
  - Including isotopy, replacing left/right duals with left/right mates
  - Monoidal categories is a special case: a 2-category with one object.

Filling in some missing diagrams from Lecture 2...

Trace (in rigid categories)

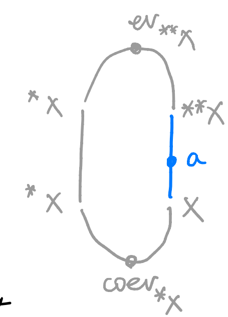
- $\mathcal{C}$ rigid category ; $X \in \mathcal{C}$
- The left trace of $a : X \to X^{**}$ is

$$Tr^L(\alpha) : \mathbb{1} \xrightarrow{coev_X} X \otimes X^* \xrightarrow{a \otimes id_{X^*}} X^{**} \otimes X^* \xrightarrow{ev_{X^*}} \mathbb{1}$$



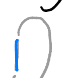
- The right trace of $a : X \to {}^{**}X$ is

$$Tr^R(\alpha) : \mathbb{1} \xrightarrow{coev_{*X}} {}^*X \otimes X \xrightarrow{id_{*X} \otimes a} {}^*X \otimes {}^{**}X \xrightarrow{ev_{**X}} \mathbb{1}$$



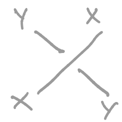
- Putting pivotal structure map gives dimension of corresp. object

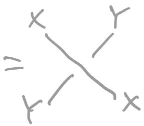
<u>Example</u> in TL_1, the generating object is self dual so

$\dim^L(\bullet) = Tr^L(\,|\,) =$  $= 1$

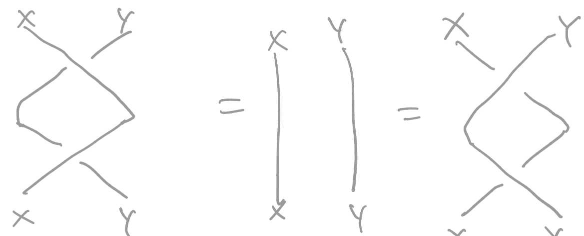
Braided Monoidal Categories

A braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ is

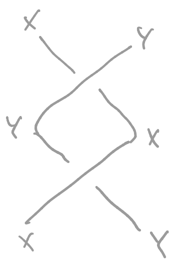


with inverse $c_{X,Y}^{-1} =$ 

This is sensible since $c_{X,Y}^{-1} \circ c_{X,Y} = id_{X \otimes Y} = c_{X,Y} \circ c_{X,Y}^{-1}$



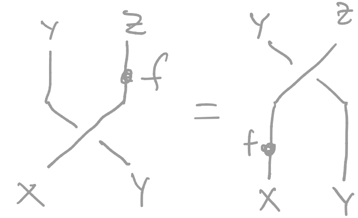
and $c_{Y,X} \circ c_{X,Y}$ is not necessarily the identity



Braided Mon. Cat. (cont.)

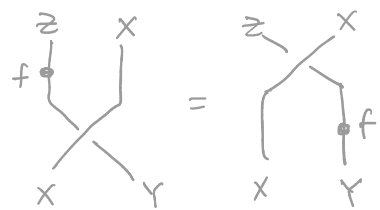
- Naturality of the braiding, for $f: X \to Z$, $g: Y \to Z$

$$\begin{tikzcd}
X \otimes Y \arrow[r, "c_{X,Y}"] \arrow[d, "f \otimes id_Y"'] & Y \otimes X \arrow[d, "id_Y \otimes f"] \\
Z \otimes Y \arrow[r, "c_{Z,Y}"'] & Y \otimes Z
\end{tikzcd}$$



"sliding dot past a crossing"

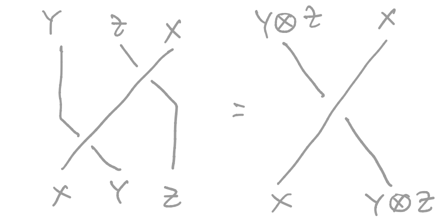
$$\begin{tikzcd}
X \otimes Y \arrow[r, "c_{X,Y}"] \arrow[d, "id_X \otimes f"'] & Y \otimes X \arrow[d, "f \otimes id_X"] \\
X \otimes Z \arrow[r, "c_{X,Z}"'] & Z \otimes X
\end{tikzcd}$$



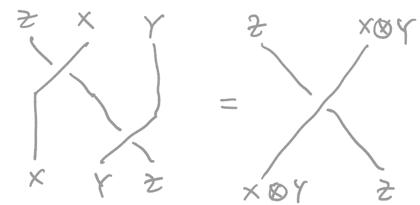
Braided Mon. Cat. (cont.)

- Hexagonal commuting diagrams

$$
\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r, "c"] \arrow[dl, equal, "\alpha"'] & (Y \otimes Z) \otimes X \arrow[dr, equal] & \\
(X \otimes Y) \otimes Z \arrow[dr, "c"'] & & & Y \otimes (Z \otimes X) \\
 & (Y \otimes X) \otimes Z \arrow[r, equal, "\alpha"'] & Y \otimes (X \otimes Z) \arrow[ur, "c"'] &
\end{tikzcd}
$$



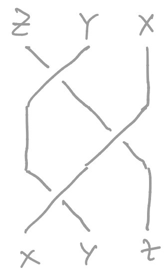
$$
\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[r, "c"] \arrow[dl, equal, "\alpha"'] & Z \otimes (X \otimes Y) \arrow[dr, equal] & \\
X \otimes (Y \otimes Z) \arrow[dr, "c"'] & & & (Z \otimes X) \otimes Y \\
 & X \otimes (Z \otimes Y) \arrow[r, equal, "\alpha"'] & (X \otimes Z) \otimes Y \arrow[ur, "c"'] &
\end{tikzcd}
$$



Braided Mon. Cat. (cont.)

- We assumed strictness so we have Yang-Baxter equation

$$(c_{Y,Z} \otimes id_X) \circ (id_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes id_Z) = (id_Z \otimes c_{X,Y}) \circ (c_{X,Z} \otimes id_Y) \circ (id_X \otimes c_{Y,Z})$$



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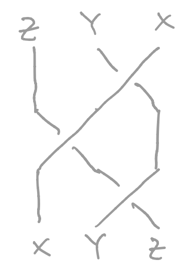
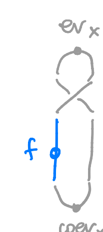


"moving a crossing under a strand"

Exercise Prove this using naturality and hexagon relations.

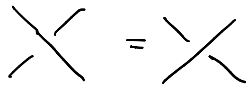
- Trace (for rigid braided), for $f: X \to X$

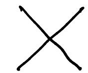
$$tr^L(f): \mathbb{1} \xrightarrow{coev_X} X \otimes X^* \xrightarrow{f \otimes X^*} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{ev_X} \mathbb{1}$$



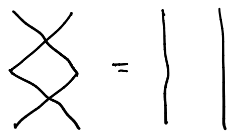
Symmetric Mon. Cat.

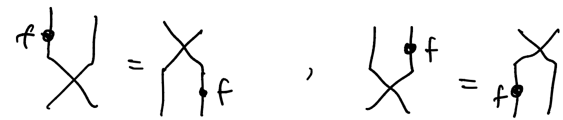
This happens when $c_{X,Y}^{-1} = c_{Y,X}$

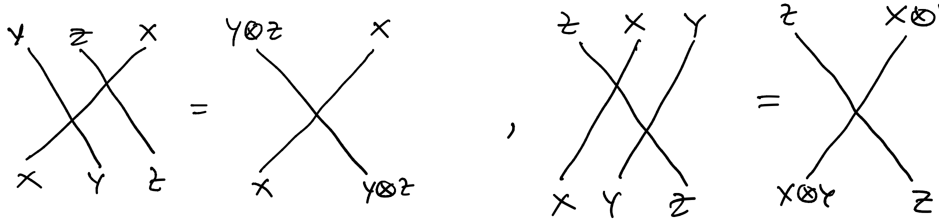


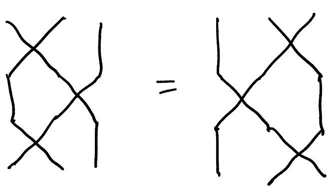
so we just write 

Then the braided category relations become

- 

- 

- 

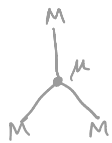
-  "braid relation"

Example

Monoid, Comonoid, Frobenius objects

DEF A <u>monoid object</u> in a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is a triple (M, μ, η) with $M \in \mathcal{C}$ an object, and morphisms

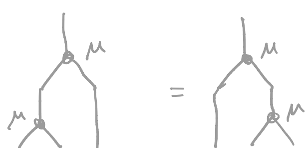
$$\mu : M \otimes M \to M \quad \text{"multiplication"}, \qquad \eta : \mathbb{1} \to M \quad \text{"unit"}$$

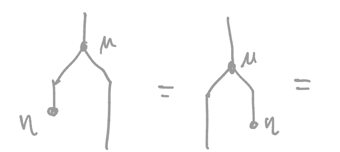


such that the following commute

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[rr, "\alpha"] \arrow[d, "\mu \otimes id_M"'] & & M \otimes (M \otimes M) \arrow[d, "id_M \otimes \mu"] \\
M \otimes M \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes M \arrow[r, "\eta \otimes id_M"] \arrow[dr, "l_M"'] & M \otimes M \arrow[d, "\mu"] & M \otimes \mathbb{1} \arrow[l, "id_M \otimes \eta"'] \arrow[dl, "r_M"] \\
& M &
\end{tikzcd}$$

(assuming strict)



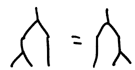


Monoid, Comonoid, Frobenius objects (cont.)

Examples

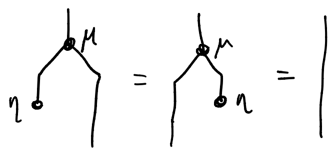
① A monoid object M in Set ($\otimes$ = cartesian prod, $\mathbb{1} = \{*\}$) is a monoid in the usual sense

- $\mu : M \times M \to M$ is an associative binary op. on M



- $\eta : \{*\} \to M$ picks an identity element from the set

and



says that this element is indeed identity of μ

② A monoid object in Cat (small categories ; $\otimes$= cartesian product, $\mathbb{1}$ = trivial 1-object category) is a (small) monoidal category

Exercise check the commutative diagrams work out

Monoid, Comonoid, Frobenius objects (cont.)

③ A monoid object in Ab (abelian groups ; $\otimes := \otimes_{\mathbb{Z}}$, $\mathbb{1} = \mathbb{Z}$) is a ring (with identity)

④ A monoid object in $Vect_k$ (f.d. k-vectorspaces ; $\otimes := \otimes_k$, $\mathbb{1} = k$)
is a unital associative algebra

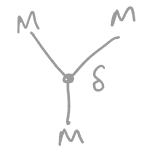
⑤ A monoid object in $End(\mathcal{C})$ (category of endofunctors of $\mathcal{C}$; $\otimes$ = horizontal composition, $\mathbb{1} = \mathrm{Id}_{\mathrm{id}_{\mathcal{C}}}$)

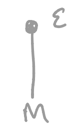
is a monad.

Monoid, Comonoid, Frobenius objects (cont.)

DEF A <u>comonoid object</u> in a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is a triple (M, δ, ε) with $M \in \mathcal{C}$ an object, and morphisms

$$\delta : M \longrightarrow M \otimes M \text{ "comultiplication"}, \qquad \varepsilon : M \longrightarrow \mathbb{1} \text{ "counit"}$$

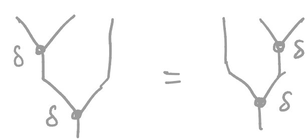


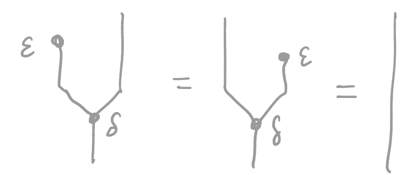


such that the following commute

$$\begin{tikzcd}
(M\otimes M)\otimes M \arrow[rr, "\alpha"] & & M\otimes(M\otimes M) \\
M\otimes M \arrow[u, "\delta\otimes id_M"] & & M\otimes M \arrow[u, "id_M\otimes\delta"'] \\
& M \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1}\otimes M & M\otimes M \arrow[l, "\varepsilon\otimes id_M"'] \arrow[r, "id_M\otimes\varepsilon"] & M\otimes\mathbb{1} \\
& M \arrow[ul, "l_M^{-1}"] \arrow[u, "\delta"'] \arrow[ur, "r_M^{-1}"'] &
\end{tikzcd}$$

(assuming strict)





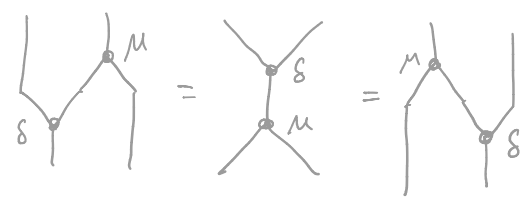
Monoid, Comonoid, Frobenius objects (cont.)

DEF A <u>Frobenius object</u> in a Monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is a quintuple $(A, \mu, \delta, \eta, \varepsilon)$ such that

- (A, μ, η) is a monoid object
- (A, δ, ε) is a comonoid object
- the following compatibility diagram commutes

$$\begin{tikzcd}
 & A \otimes A \arrow[dl, "\delta \otimes id_A"'] \arrow[d, "\mu"] \arrow[dr, "id_A \otimes \delta"] & \\
(A \otimes A) \otimes A \arrow[d, "\alpha"'] & A \arrow[dd, "\delta"] & A \otimes (A \otimes A) \arrow[d, "\alpha"] \\
A \otimes (A \otimes A) \arrow[dr, "id_A \otimes \mu"'] & & (A \otimes A) \otimes A \arrow[dl, "\mu \otimes id_A"] \\
 & A \otimes A &
\end{tikzcd}$$

(assuming strict)



<u>Example</u> In TL_1, $\bullet \otimes \bullet$ is a Frobenius object with

$\mu =$  $\eta =$ 

$\delta =$  $\varepsilon =$ 

<u>Exercise</u>: Check the relations!