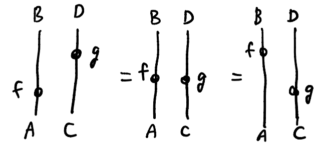
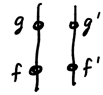
Exercises (lecture 3)

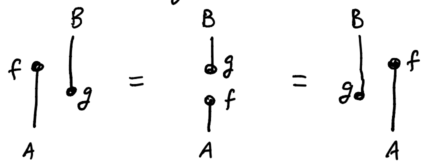
① Write the commutative diagram to show the interchange law holds.



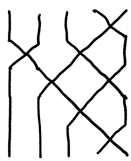
Hint: use bifunctoriality of $\otimes$

② Show that the two interpretations of  agree.

③ Prove the following relation (assuming strict)



④ In the symmetric category Sym, write the following as a diagram corresponding to the disjoint cycle decomposition in S_5



⑤ Show that TL_2 is rigid and pivotal. (Draw the defining morphisms and relations)

⑥ Convince yourself that all morphisms in TL_1 are cyclic

⑦ Consider the monoidal category generated by a $\mathbb{C}$-vector space V with basis $\{e_1, e_2\}$, and morphisms

$$n : V \otimes V \to \mathbb{C} \qquad , \qquad u : \mathbb{C} \to V \otimes V$$
$$e_1 \otimes e_1 \mapsto 0 \qquad\qquad 1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$$
$$e_2 \otimes e_2 \mapsto 0$$
$$e_1 \otimes e_2 \mapsto -1$$
$$e_2 \otimes e_1 \mapsto 1$$

Show that this is equivalent to TL_{-2} by checking the zig-zag relation with

$$\cap \leftrightarrow n, \quad \cup \leftrightarrow u$$

Additionally, assuming V is the standard rep of $SL(2,\mathbb{C})$ (action is matrix multiplication) and $\mathbb{C}$ is trivial representation, show n and u are $SL(2,\mathbb{C})$ equivariant

⑧ (Hard) Find all the primitive idempotents of TL_{-2}
- These are all the indecomposible representations of $SL(2,\mathbb{C})$

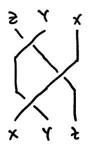
If this is too hard, start by finding idempotents in

$$\mathrm{End}(\bullet), \quad \mathrm{End}(\bullet \otimes \bullet), \quad \mathrm{End}(\bullet \otimes \bullet \otimes \bullet)$$

⑨ Recall diagrams for the 2-category Cat.
- Dual objects in monoidal categories correspond to adjoint functors.

For adjoint functors
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd},$$
what does the isomorphism $\mathrm{Hom}_{\mathcal{D}}(FX, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, GY)$ look like with diagrams?

⑩ Show the Yang-Baxter equation



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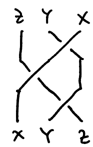

diagramatically using naturality and hexagon relations

⑪ Check that monoid objects in Cat are actually monoidal categories.

(Show the aspects of definition of monoidal cat. follow from that of the monoid object structure)

⑫ Check that in TL_1, $\bullet \otimes \bullet$ is a Frobenius object with

$\mu =$  , $\eta =$ 

$\delta =$  , $\varepsilon =$ 

⑬ (From last week) Draw the Drinfeld morphism

$$u_X : X \xrightarrow{X \otimes \mathrm{coev}_{X^*}} X \otimes X^* \otimes X^{**} \xrightarrow{c_{X,X^*} \otimes X^{**}} X^* \otimes X \otimes X^{**} \xrightarrow{ev_X \otimes X^{**}} X^{**}$$

and the relation $u_{X\otimes Y} \circ c_{Y,X} \circ c_{X,Y} = u_X \otimes u_Y$