Data analysis and Quantum knot invariants

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joint with
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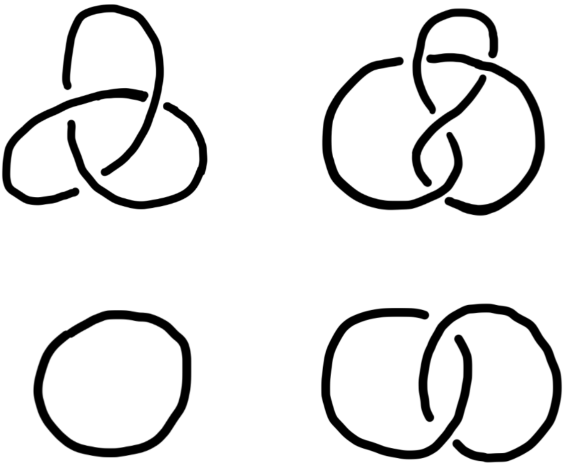
Outline

1. Introduction: Knots, Links and Invariants
2. Why big data?
3. Jones polynomial and the unknot

Introduction

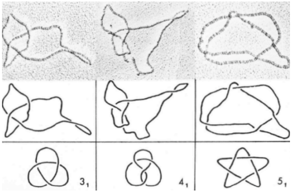
Knots and Links



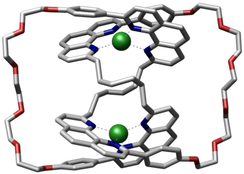
Introduction

Motivation

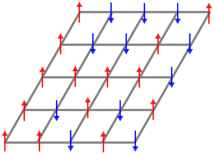
DNA



Molecules



Statistical Mechanics



Invariants

Question: Is this the unknot?

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Mathematical tool: *invariants*,
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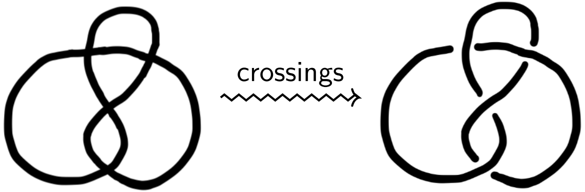
We focus on polynomial invariants in this talk,
e.g. the Jones polynomial of the above knot is 1.

Planar diagram presentation



4-regular planar graph
“shadow”

knot or link

The Jones polynomial

Jones' polynomial (up to normalisation)

$$\langle \cdot \rangle : \text{Links} \to \mathbb{Z}[q^{\pm 1}]$$

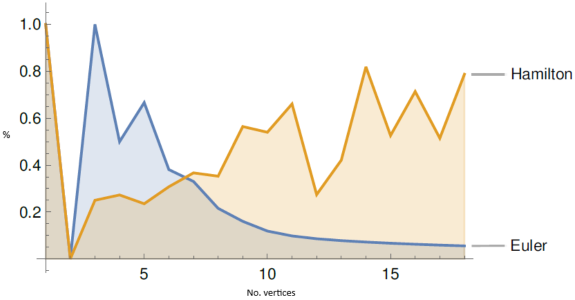
$$\langle \bigcirc \rangle = 1$$
$$\left\langle \times \right\rangle = q \left\langle)\,(\right\rangle + q^{-1} \left\langle \asymp \right\rangle$$
$$\langle \bigcirc \sqcup \mathcal{L} \rangle = (-q^2 - q^{-2}) \langle \mathcal{L} \rangle$$

Why big data?

Analysing lots of data gives evidence about asymptotic/average behaviours

Analogy: Random graph theory

- a random graph is almost surely connected
- a random graph almost surely has a Hamiltonian cycle (Pósa 1976)



- (many more)

Knots as big data

- Connected sum $K_1 \# K_2 = K$
- Prime knots are prime wrt. connect sum $\#$ and unknot $\bigcirc$ as the unit.
- Jones poly. respects $\#$: $J(K_1 \# K_2) = J(K_1) \cdot J(K_2)$

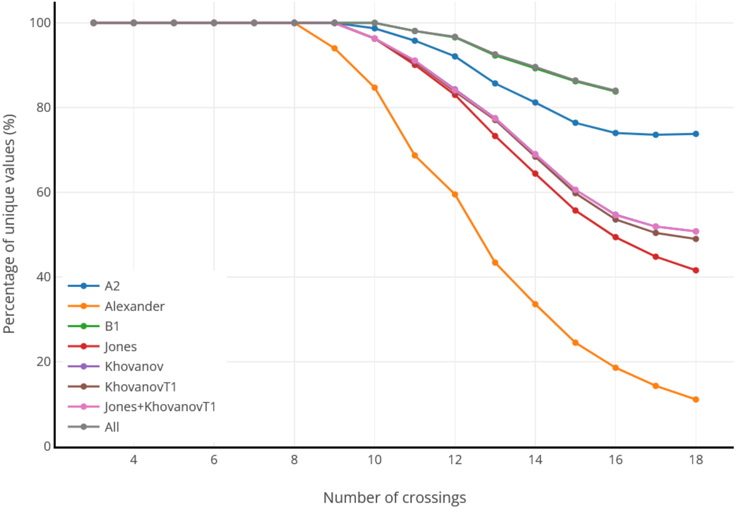
Prime knots

x no.	3	4	5	6	7	8	9	10	11	12
count	1	1	2	3	7	21	49	165	552	2176

13	14	15	16	17
9988	46972	253293	1388705	8053393

18	19	20
48266466	294130458	1847319428

Knots as big data

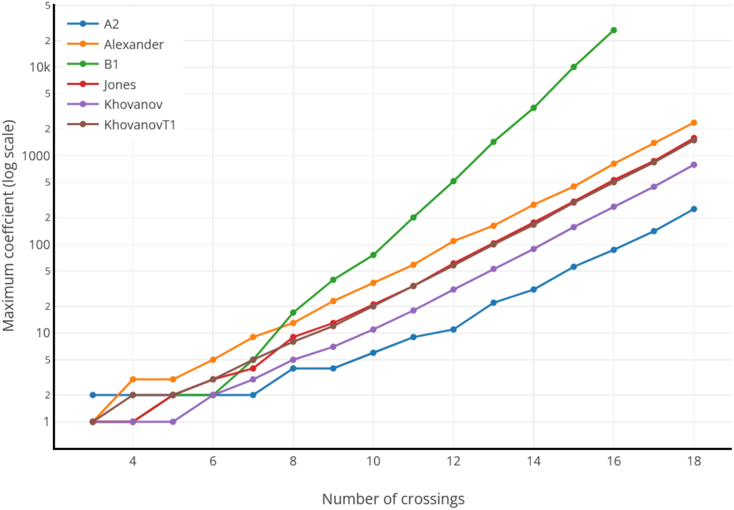


Theorem (Tubbenhauer–Z. 2025 (website))

"All quantum invariants" exponentially decay to 0% uniqueness.

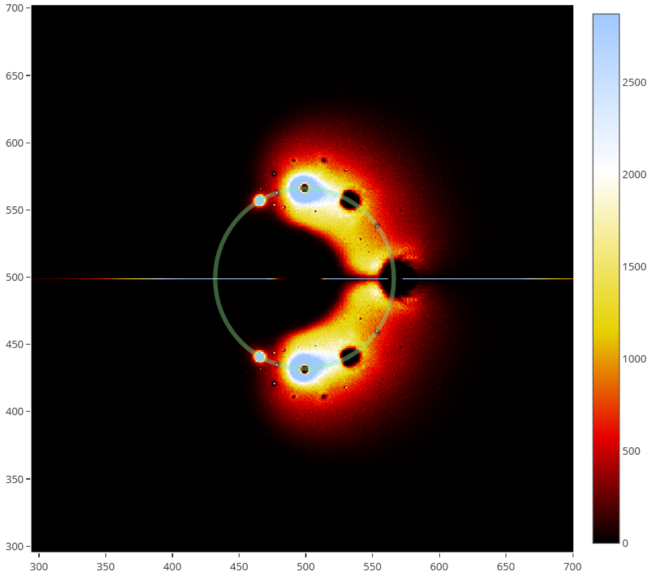
Knots as big data

Maximum coefficient



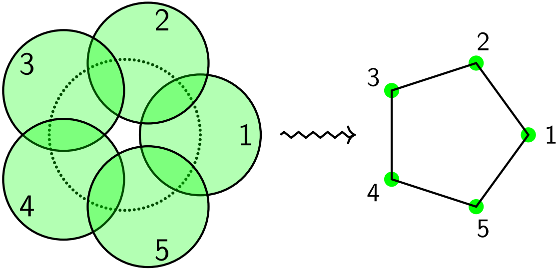
Knots as big data

Roots of Jones polynomials



Knots as big data

Topological data analysis: Ball mapper (Dłotko 2019)



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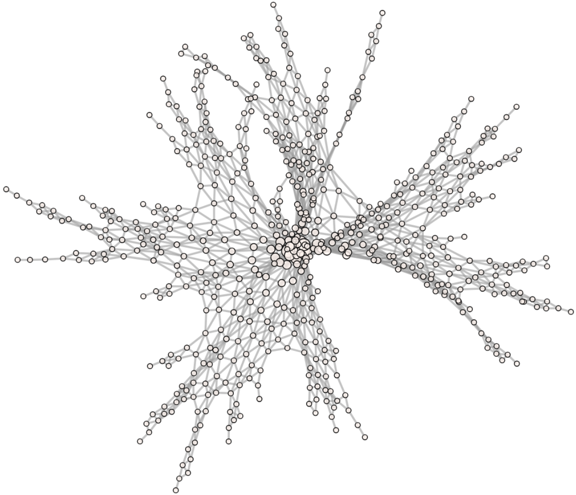

Jones up to 16 crossings (interactive plot)

Knots as big data

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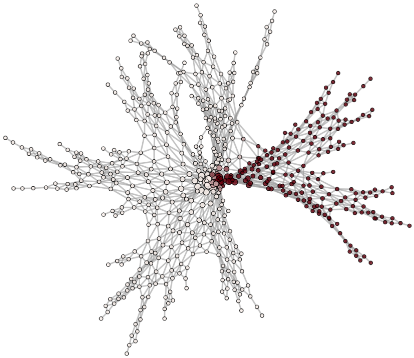
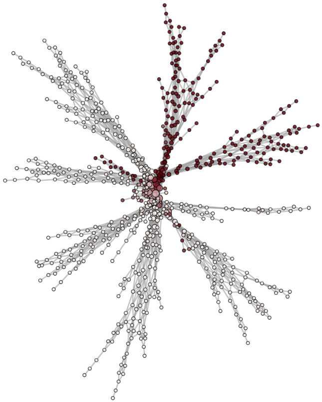


Jones up to 16 crossings



Khovanov up to 16 crossings

An age old question

Does the Jones polynomial detect the unknot?
i.e. is $J(K) = 1 \iff K = \bigcirc$?

Existing data

- true for all knots up to 18 crossings;
- true for experiments run over random knots;
- neural networks unable to learn non-trivial knots with trivial Jones polynomial.

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Neural networks are good at finding counterexamples...
but *how can they be used to prove something?*

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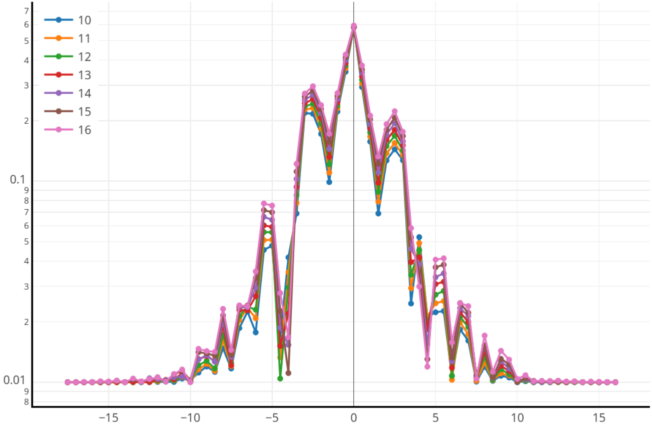

Average Jones polynomials

An age old question

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Idea: track asymptotic behaviour of counterexamples mod p

- Cooper–Long (1990s), Gibson–Williamson–Yacobi (2023) showed unfaithfulness of 4-strand Burau representation mod $2, 3, 5$.
- This is equivalent to the Jones question, but braid presentations of knots are inefficient.

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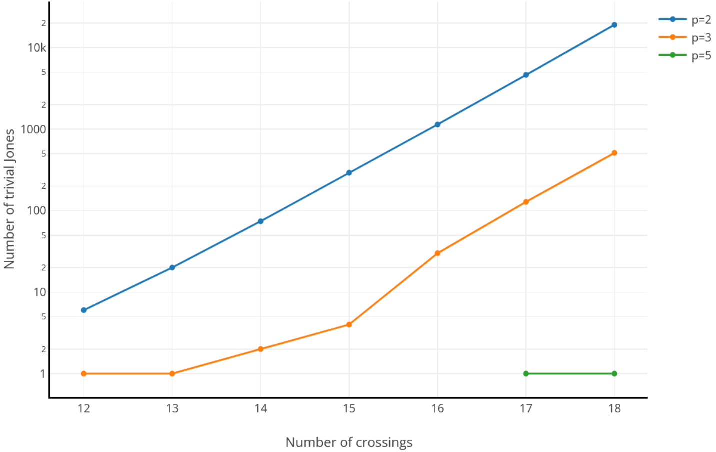
Ongoing: reduce Jones mod p directly on knots (*planar diagrams*).



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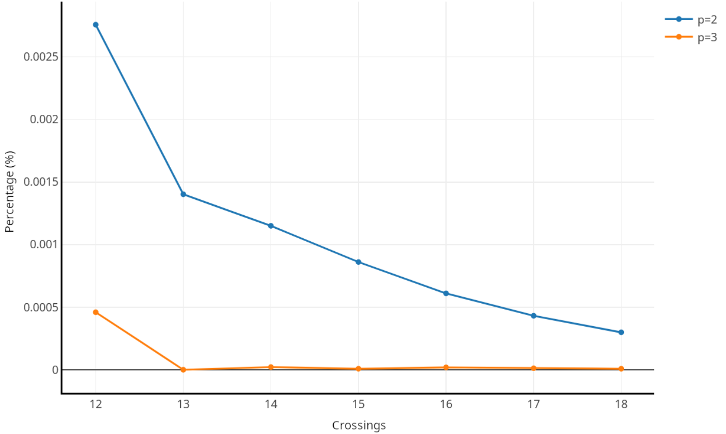

Thank you. Questions?