Aims Explain the diagrammatics of LV category and give a sketch proof of equivalence

Outline (only for $SL(2,\mathbb{R})$)

① Soergel Bimodules recap and diagrammatics
- Soergel bimodules
- One colour diagrammatics
- Graded rk, Light leaves basis
- Equivalence theorem

② Lusztig - Vogan Category for $SL(2,\mathbb{R})$
- Definition and indecomposable objects
- Diagrammatics
- Proof equivalence (sketch)

① Soergel Bimodules recap

- We do it just for sl_2 ie. $W = S_2$, $S = \{s\}$
- Let $R = \mathbb{R}[\alpha_s]$ with action of W by $s(\alpha_s) = -\alpha_s$ extending linearly & multiplicatively
  - The fixed ring $R^S := \{r \in R : s(r) = r\} = \mathbb{R}[\alpha_s^2]$
  - R is graded s.t. α_s is degree 2.
  - There is an R^S-linear operator (Demazure)

$$\partial_s : R \longrightarrow R^S(-2)$$
$$f \longmapsto \frac{f - s(f)}{\alpha_s}$$

This gives a splitting $R \cong R^S \oplus R^S\alpha_s$ as R^S-gbim

$$f \mapsto \left(\partial_s\left(f\tfrac{\alpha_s}{2}\right), \partial_s(f)\right)$$
$$g + h\tfrac{\alpha_s}{2} \leftarrow (g,h)$$

- Bott Samelson bimodules are generated by

$$B_s := R \underset{R^S}{\otimes} R(1)$$
$$BS(s_1, \dots, s_n) := B_{s_1} \otimes \cdots \otimes B_{s_n}$$

the category of Bott-Samelson Bimodules BSBim is the full subcategory of (R,R)-gbim where the objects are BS-bimodules
- clearly is monoidal

Objects? $\mathbb{1} = R$, B_s, $B_s \otimes B_s = R \underset{R^S}{\otimes} R(1) \otimes R \underset{R^S}{\otimes} R(1)$, etc.

R as an R^S-bim

$$\cong R \underset{R^S}{\otimes} R \underset{R^S}{\otimes} R(2)$$
$$\cong R \underset{R^S}{\otimes} (R^S \oplus R^S(-2)) \underset{R^S}{\otimes} R(2)$$
$$\cong \left(R \underset{R^S}{\otimes} R(2)\right) \oplus \left(R \underset{R^S}{\otimes} R\right)$$
$$= B_s(1) \oplus B_s(-1)$$

- Soergel bimodules are direct summands of direct sums of Bott-Samelson Bimodules
  - SBim is the full subcategory of (R,R)-gbim with soergel bimodules as objects.
  - is a monoidal category, additive, R-linear, idempotent complete

Objects? We have all of BSBim, and $B_s \otimes B_s = B_s(1) \oplus B_s(-1)$ means we have no more.
Morphisms?

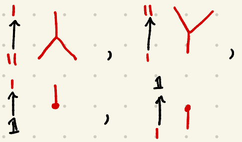
· Diagrammatics

$\mathcal{DH}$ "diagrammatic Hecke category"

objects: $\mathbb{1}$, $|$, $|\otimes|$, $|\otimes|\otimes|$, etc.

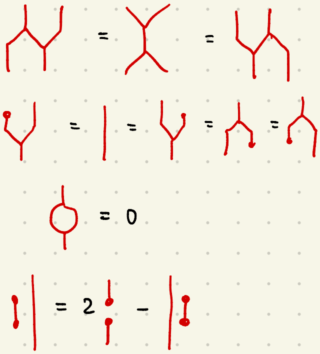
(note we identify the generating object $|$ with it's identity morphism. This makes thing easier later)

morphisms: $id_{\mathbb{1}}$ blank diagram,



(all compositions and tensor products of these)

local relations:



(It can be proven that we have arbitrary isotopy with these relations so the first two may be written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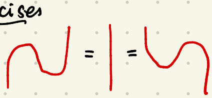
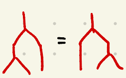
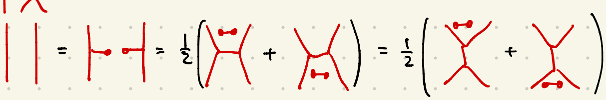
Note some places use polynomial boxes. We treat  as α_s, so R will act on these morphisms by tensoring by a linear combination of appropriate diagrams

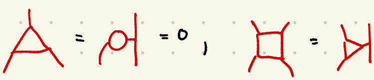
eg $2+\alpha_s-3\alpha_s^2$ acts by $-\otimes(2id_{\mathbb{1}}$ +  $- 3$ $)$

Exercises

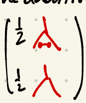
• 

•  and it's vertical reflection

• 

•  etc.

• Extending to the additive closure, we have inverses

 and  check!

Light leaves Basis

- each $\mathrm{Hom}(1^{\otimes n}, 1^{\otimes m})$ has finite graded rank over R or generally over $\mathbb{Z}[\ ]$
- There is a basis called the light leaves basis (Libedinsky)

We give the diagrammatic construction for them (Elias-Williamson)

__Example__ $\mathrm{Hom}(1^{\otimes 3}, 1^{\otimes 2})$

Do the following for both domain & codomain

- think of $1^{\otimes 3}$ as an expression in S_2 (s,s,s)
- consider all possible subexpressions of (s,s,s)

  ie. all binary strings corresponding to choice of which ones to include

  $(0,0,0) \rightsquigarrow 111 = 1$, $(0,0,1) \rightsquigarrow 11s = s$, $(0,1,1) \rightsquigarrow 1ss = 1$ etc.
- For each subexpression iterate through and decide whether it should be labelled U for up or D for down

eg

i	partial sub.	potential	reality
0	$\varnothing$	1	1
1	(1)	s	s
2	(1,0)	ss=1	s1 = s
3	(1,0,1)	ss=1	ss = 1

Label is U when multiplying the next term to the current partial subexpression the element increases in Bruhat order (for us just think $1 < s$) and D otherwise.

- Construct morphism given subexpression + U/D-sequence

U1 U0 D1 D0

intuition: starting with the full expression at the bottom, visually construct the subexpression

- Eg. for $1^{\otimes 3}$,

000	001	010	011	100	101	110	111
UUU	UUU	UUD	UUD	UDD	UDD	UDU	UDU

Notice the codomain of these are the subexpressions

These are called light leaves.

- The basis of Hom spaces are called double light leaves, where we compose light leaves and their vertical reflection (with the subexpression in the middle)

- eg for $\mathrm{Hom}(1^{\otimes 3}, 1^{\otimes 2})$ we also do (s,s) :

00	01	10	11
UU	UU	UD	UD

So the double leaves basis:

- common sub (s) :

3 1 1 -1 -1 -3 1 -1

- common sub $\varnothing$:

5 3 3 1 1 -1 3 1

We can check this aligns with the Soergel hom formula that tells us the graded ranks.

$$\mathrm{grrk}\,\mathrm{Hom}^\bullet(B_s^{\otimes 3}, B_s^{\otimes 2}) = (b_s^3, b_s^2) = (b_s^5, 1)$$
$$= \varepsilon((\delta_s + v)^3)$$
$$= \varepsilon(\delta_s^5 + 5\delta_s^4 v + 10\delta_s^3 v^2 + 10\delta_s^2 v^3 + 5\delta_s v^4 + v^5)$$
$$= ((v^{-1}-v)^2+2)(v^{-1}-v) + 5v((v^{-1}-v)^2+1)$$
$$+ 10v^2(v^{-1}-v) + 10v^3 + v^5$$
$$= v^{-1}-v + (v^{-1}-v+5v)((v^{-1}-v)^2+1) + 10v^2(v^{-1}-v) + 10v^3 + v^5$$
$$= v^{-1}-v + (v^{-1}+4v)(v^{-2}-2+v^2+1) + 10v - \cancel{10v^3} + \cancel{10v^3} + v^5$$
$$= \cancel{v^{-1}} + 9v + v^5 + v^{-3} - \cancel{v^{-1}} + v + 4v^{-1} - 4v + 4v^3$$
$$= v^5 + 4v^3 + 6v + 4v^{-1} + v^{-3}$$

$$\delta_s^2 = (v^{-1}-v)\delta_s + 1$$
$$\delta_s^3 = (v^{-1}-v)\delta_s^2 + \delta_s$$
$$= (v^{-1}-v)^2\delta_s + (v^{-1}-v) + \delta_s$$
$$\delta_s^4 = ((v^{-1}-v)^2+1)(v^{-1}-v)\delta_s$$
$$+ ((v^{-1}-v)^2+1)$$
$$+ (v^{-1}-v)\delta_s$$
$$= ((v^{-1}-v)^2+2)(v^{-1}-v)\delta_s$$
$$+ ((v^{-1}-v)^2+1)$$
$$\delta_s^5 = ((v^{-1}-v)^2+2)(v^{-1}-v)^2\delta_s$$
$$+ ((v^{-1}-v)^2+2)(v^{-1}-v)$$
$$+ ((v^{-1}-v)^2+1)\delta_s$$

THM (Elias–Williamson)

The following functor is an equivalence of $\mathbb{Z}$[]-linear monoidal categories.

$$\mathcal{DH} \longrightarrow \mathrm{BSBim}$$

ob: $\mathbb{1} \longmapsto R$

 $\longmapsto B_s$

mor:  $\longmapsto \mu : B_s \otimes B_s \longrightarrow B_s$

$f \otimes_s g \otimes_s h \longmapsto \partial_s(g)\, f \otimes h$

 $\longmapsto \delta : B_s \longrightarrow B_s \otimes B_s$

$f \otimes g \longmapsto f \otimes 1 \otimes g$

 $\longmapsto \eta : R \longrightarrow B_s$

$1 \longmapsto \frac{1}{2}(1 \otimes \alpha_s + \alpha_s \otimes 1)$

 $\longmapsto \varepsilon : B_s \longrightarrow R$

$f \otimes g \longmapsto fg$

We won't prove this.

② Lusztig-Vogan Category for $SL(2,\mathbb{R})$

- Associated to $W = S_2$, $S = \langle s \rangle$ and $W_K = \langle 1 \rangle$

- The LV-category with this data is the full subcategory of (R^{W_K}, R)-gbim with objects given by $\langle R_w \cdot SBim(W,S) : w \in W_K \rangle_{\oplus, \ominus, (1)}$

  dir sums, dir summands, grading shifts

  - We write $\mathcal{N}$ for this, with only degree 0 morphisms

  - Write $\tilde{\mathcal{N}}$ for the similar category with objects $\langle R_w \cdot BSBim(W,S) : w \in W_K \rangle$ and graded Homs

  - Think of $\tilde{\mathcal{N}}$ like BSBim and $\mathcal{N}$ like SBim.
  - By construction $\tilde{\mathcal{N}}$ is a module cat over BSBim and $\mathcal{N}$ is a module cat over SBim

- Indecomposable objects in $\mathcal{N}$:

  <u>LEMMA (Extension of LEM 4.34 in SBim Book)</u>

  If M is a graded (R^{W_K}, R)-bimodule generated by homogeneous element $m \in M$. Then M is indecomposable.

  <u>Proof</u> let m have degree d. If $M = L \oplus N$ then $M^d = L^d \oplus N^d$. Since $(R^K)^0 = R^0 = \mathbb{R}$, $M^d = \mathbb{R}m$, so we may assume $m \in L$. Then $M = R^K \cdot m \cdot R = L$ so $N = 0$. $\square$

  R is indecomposable (gen. by 1)

  R_s is indecomposable (gen. by 1)

  $R \otimes B_s = B_s$ is indecomposable (gen. by $1 \otimes_s 1$)

  $$\begin{aligned} R_s \otimes B_s &= R_s \otimes R \otimes_{R^s} R(1) \\ &\simeq R_s \otimes_{R^s} R(1) \\ &\simeq R \otimes R(1) \\ &= B_s \text{ is indecomposable} \end{aligned}$$

  <u>Note</u> the isomorphism
  $$\begin{aligned} R_s \otimes B_s &\xrightarrow{\sim} B_s \\ f \otimes g \otimes_s h &\longmapsto f s(g) 1 \otimes_s h \\ f \otimes 1 \otimes_s h &\longleftarrow f \otimes_s h \end{aligned}$$

  - There are no more objects of interest

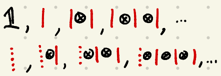
- Morphisms ?

  - we skip the motivation for the construction, ask me if interested

Diagrammatics of $\tilde{N}$

$\mathcal{D}\tilde{N}$ "Diagrammatic LV-category"
w/ action of $\mathcal{DH}$

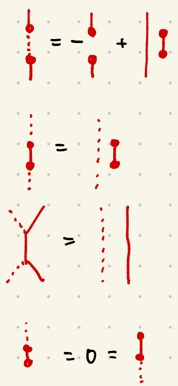
objects $\mathbb{1}$,



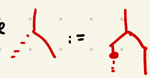
morphisms $id_{\mathbb{1}}$,

(w/ tensor of $\mathcal{DH}$ on the right, and compositions)

local relations

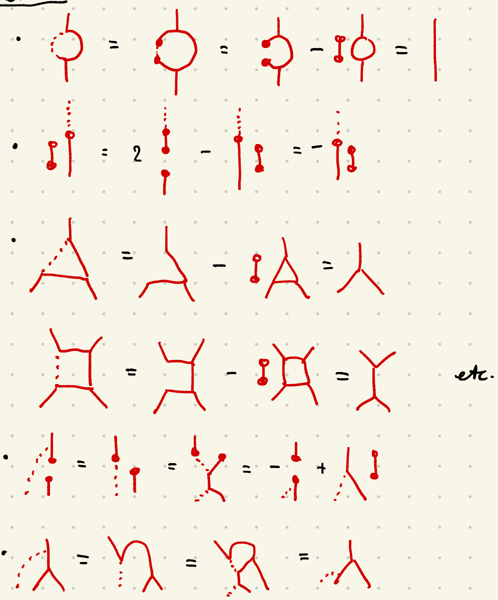


where



Note by how this category is generated, the dotted line must always be on the left of everything
So we allow arbitrary isotopy as long as ⋮ is never on the right of anything (follows from isotopy of $\mathcal{DH}$ because we can't bend ⋮)

Exercises



THEOREM ①

Let $F: \mathcal{D}\tilde{N} \to \tilde{N}$ be the additive R-linear right $\mathcal{DH}$-module category functor defined on diagrammatic generators s.t.

objects: $\mathbb{1} \longmapsto R$

$\longmapsto R_s$

morphisms: $\longmapsto \eta_s : R_s \to B_s$

$1 \longmapsto \frac{1}{2}(1 \otimes \alpha_s - \alpha_s \otimes 1)$

$\longmapsto \varepsilon_s : B_s \to R_s$

$f \otimes_s g \longmapsto f s(g)$

[notice there are analogues of ,]

Then F is an equiv. of additive R-linear right $\mathcal{DH}$-module categories.

Proof Sketch

- It can be checked that η_s and ε_s are R-bimodule homs.
- We can extend F to a $\mathcal{DH}$-module cat. functor by extending by $\mathcal{DH} \simeq \mathbb{BS}Bim$
- We can check F is well defined by preserving relations.

  In particular $F(\curlyvee)$, $F(\curlywedge)$ are the isomorphisms defining $R_s \otimes B_s \simeq B_s$
- Due to the isomorphism $\otimes |^{\otimes n} \simeq |^{\otimes n}$ the hom spaces have dim equal to if we forget about
- We have basis

$$\left\{ \phi \in \mathbb{LL} : \boxed{\phi} \right\} \quad \text{for } \mathrm{Hom}(\otimes|^{\otimes n}, |^{\otimes m})$$

and similarly for other homs.
  - Spanning:
    ① remove all internal
    ② move all connection to | to the bottom left (using a previous rel.)
    ③ move all floating diagrams to the right
    ④ remove all diagrams where is connected to floating diagram
    ⑤ Result should be right $\mathbb{Z}[\,]$ lin. comb. of our set of morphisms
  - linearly indep: compose w/ isomorphisms to get lin. comb of double leaves then the coeff are 0 bc. double leaves are lin. indep.
- Since $\curlywedge$, $\curlyvee$ are degree 0, this new basis matches the dimensions of the Hom-space.

It can be checked that this gets sent to a linearly independent set by F, so it is full and faithful.