The Diagrammatic Lusztig–Vogan Category

for $\mathrm{SL}(2, \mathbb{R})$

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Australian Mathematical Sciences
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Outline

1. Context and Motivation
2. Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$
3. Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$
4. Further work

Context and Motivation

Complex (simply-connected) Lie groups

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BGG
Category $\mathcal{O}$

Vermas, Simples

Context and Motivation

Complex (simply-connected) Lie groups

BGG Category $\mathcal{O}$ $\rightsquigarrow$ Hecke algebra $\mathcal{H}$

Vermas, Simples

$\{\delta_s\}, \{b_s\}$

Context and Motivation

Complex (simply-connected) Lie groups

BGG Category $\mathcal{O}$ ⇝ Hecke algebra $\mathcal{H}$ ⇝ (categorification) Soergel bimodules $\mathbb{S}$Bim

Vermas, Simples

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Real (reductive) Lie groups

Context and Motivation

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Real (reductive) Lie groups

Category of admissible reps.

Principal series', Simples

Context and Motivation

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- BGG Category $\mathcal{O}$: Vermas, Simples
- Hecke algebra $\mathcal{H}$: $\{\delta_s\}, \{b_s\}$

Real (reductive) Lie groups

Category of admissible reps. ⇝ LV-module $M \curvearrowleft \mathcal{H}$

- Category of admissible reps.: Principal series', Simples
- LV-module: $\{\delta\}, \{C_\delta\}$

Context and Motivation

Complex (simply-connected) Lie groups

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Real (reductive) Lie groups

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Principal series', Simples

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Context and Motivation

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- Category of admissible reps.: Principal series', Simples
- LV-module: $\{\delta\}, \{C_\delta\}$

Context and Motivation

Why diagrammatics?

- Diagrammatic methods enabled Elias & Williamson (2014) to give an algebraic proof of the Kazhdan–Lusztig conjecture (1979). This is more general than can be proved geometrically.
- Provided intuition for Williamson (2017) to discover counterexamples to Lusztig's conjecture (1980).
- $\mathcal{D}\,\mathbb{S}\mathrm{Bim}$ can be studied in contexts where $\mathbb{S}\mathrm{Bim}$ is not well-behaved, for example in fields of characteristic p.

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

The diagrammatic Hecke category $\mathcal{DH}$ (for $\mathfrak{sl}_2\mathbb{C}$) is a $\mathbb{Z}[\frac{1}{2}]$-linear monoidal category defined as follows.

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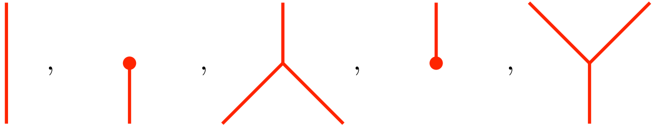
- *Morphisms*: generated by

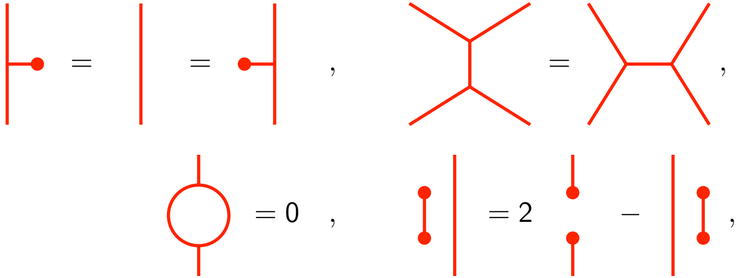


under relations...

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

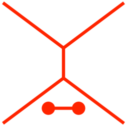
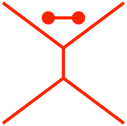
Relations:



and arbitrary planar isotopy.

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$\frac{1}{2}$  $+\ \frac{1}{2}$ 

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

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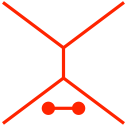
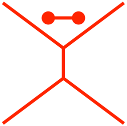
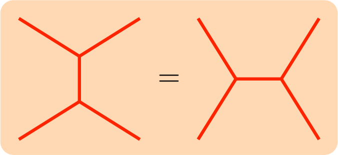
$\frac{1}{2}$  $+\frac{1}{2}$ 



Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

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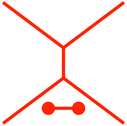
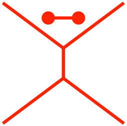
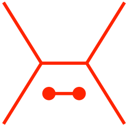
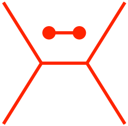
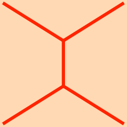
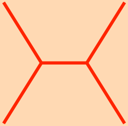
$\frac{1}{2}$  $+\ \frac{1}{2}$  $=\ \frac{1}{2}$  $+\ \frac{1}{2}$ 

 $=$ 

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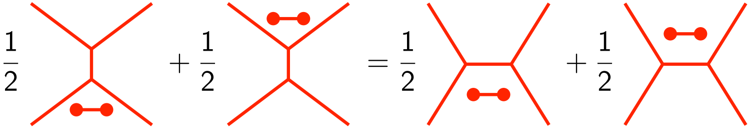

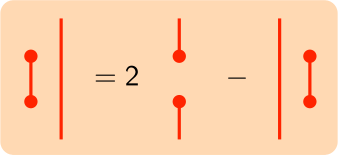

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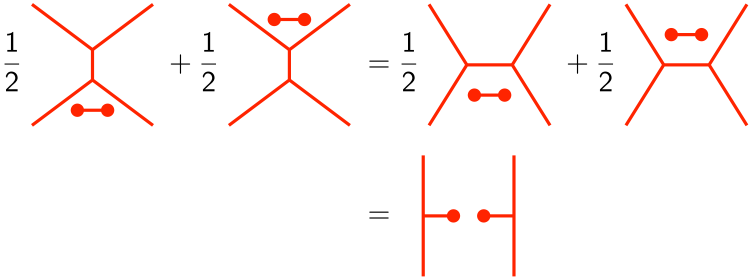

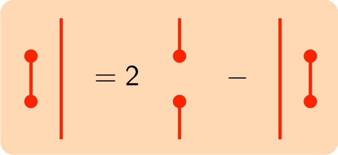

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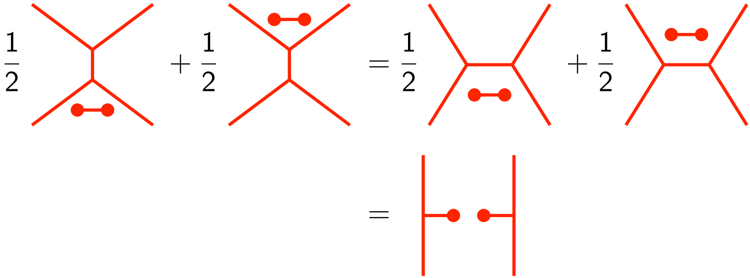

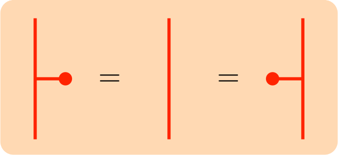

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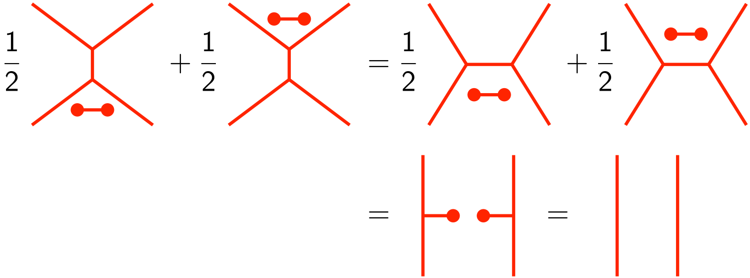

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Theorem (Elias–Khovanov, 2010[1])

The Karoubi envelope of $\mathcal{DH}$ is equivalent to the category of Soergel Bimodules $\mathbb{S}\mathrm{Bim}$ *for* $\mathfrak{sl}_2(\mathbb{C})$ *as graded additive* $\mathbb{R}$*-linear monoidal categories.*

[1] Ben Elias and Mikhail Khovanov. "Diagrammatics for Soergel categories". In: *Int. J. Math. Math. Sci.* (2010), Art. ID 978635, 58.

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Definition

The diagrammatic LV-category $\mathcal{D}\tilde{\mathcal{N}}^0$ (for $\mathrm{SL}(2,\mathbb{R})$) is a $\mathbb{Z}[\frac{1}{2}]$-linear right module category over $\mathcal{DH}$ defined as follows. The right action by $\mathcal{DH}$ is right concatenation.

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- *Objects*: generated by $\mathbb{1}$ and $\circ$
  i.e. $\mathbb{1}$, $\bullet$, $\bullet\bullet$, $\bullet\bullet\bullet$, $\bullet\bullet\bullet\bullet$, ... and $\circ$, $\circ\bullet$, $\circ\bullet\bullet$, ...

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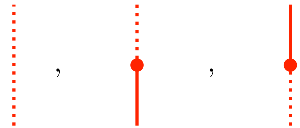
  i.e. $\mathbb{1}$, ●, ●●, ●●●, ●●●●, ... and ○, ○●, ○●●, ...
- *Morphisms*: generated by

$\mathrm{id}_{\mathbb{1}}$, 

under relations...

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Definition

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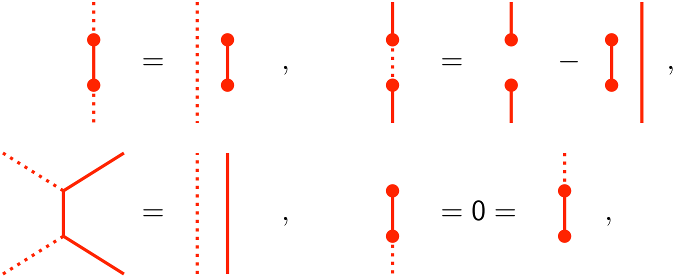

and arbitrary planar isotopy while dotted red lines never appear right of any red.

Remark

This diagrammatic definition is not entirely new: Elias–Williamson[2] had constructed very similar diagrammatics for localisation of Soergel bimodules. The LV-category is a subcategory of this, with restrained objects, morphisms and relations.

[2] Ben Elias and Geordie Williamson. "Soergel calculus". In: *Represent. Theory* 20 (2016), pp. 295–374.

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2, \mathbb{R})$

Example

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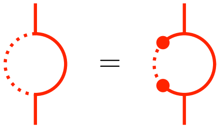

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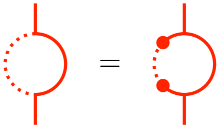

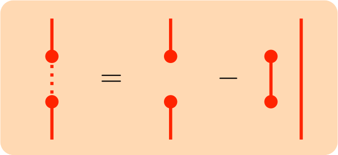

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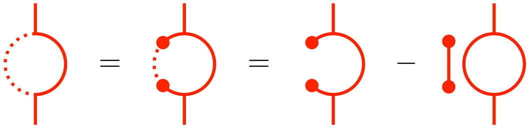
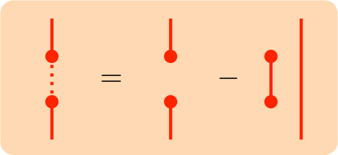
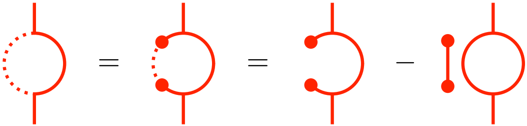




Diagrammatic Lusztig–Vogan category for $\mathsf{SL}(2,\mathbb{R})$

Example



Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

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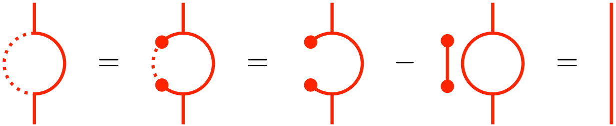


Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

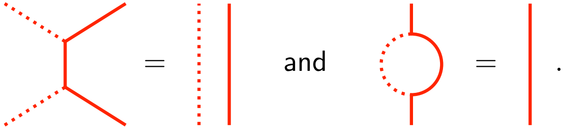
Theorem (Z.)

The Karoubi envelope of $\mathcal{D}\tilde{\mathcal{N}}^0$ is equivalent to the Lusztig–Vogan category $\mathcal{N}^0$ for $\mathrm{SL}(2,\mathbb{R})$ as graded additive $\mathbb{R}$-linear right module categories over $\mathbb{S}\mathrm{Bim}$ (for $\mathfrak{sl}_2\mathbb{C}$).

Diagrammatic Lusztig–Vogan category for $\mathrm{SL}(2,\mathbb{R})$

Proof idea

The backbone of the proof is the isomorphism $\bullet \simeq \circ\bullet$, given by the relations



This reflects an isomorphism on the level of modules that allows us to construct a basis of morphisms from an existing one in $\mathcal{DH}$.

Further Work

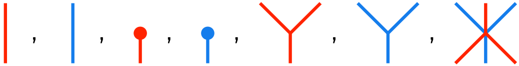
- There is an isomorphism $\mathrm{SL}(2,\mathbb{R}) \simeq \mathrm{SU}(1,1)$, so a natural next step is to consider $\mathrm{SU}(2,1)$. The aim is to define diagrammatics for the infinite family $\mathrm{SU}(n,1)$.
- Understand the dimension of the general morphism spaces at the level of the Lusztig–Vogan modules.

Extras

A Glimpse of $\mathrm{SU}(2,1)$

$\mathcal{DH}$ for $\mathfrak{sl}_3\mathbb{C}$

- *Objects:* generated by $\mathbb{1}, {\color{red}\bullet}, {\color{blue}\bullet}$.
- *Morphisms:* generated by



  and their vertical reflections. Taken up to arbitrary isotopy.
- *Relations:* Same one-colour relations. New two-colour relations (omitted).

$\mathcal{D}\tilde{\mathcal{N}}^0$ for $\mathsf{SU}(2,1)$

- Right module category over $\mathcal{DH}$ for $\mathfrak{sl}_3\mathbb{C}$.

Extras

A Glimpse of $\mathsf{SU}(2,1)$

$\mathcal{D}\tilde{\mathcal{N}}^0$ for $\mathsf{SU}(2,1)$

- Right module category over $\mathcal{DH}$ for $\mathfrak{sl}_3\mathbb{C}$.
- *Objects:* generated by $\mathbb{1}, \circ, \circ\circ$.

Extras

A Glimpse of SU(2, 1)

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- *Objects:* generated by $\mathbb{1}$, ○, ○○.
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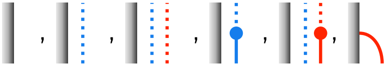


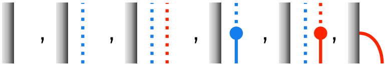
and their vertical reflections. Taken up to isotopy as long as morphisms stay within the category.

Extras

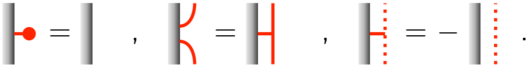
A Glimpse of SU(2, 1)

$\mathcal{D}\tilde{\mathcal{N}}^0$ for SU(2, 1)

- Right module category over $\mathcal{DH}$ for $\mathfrak{sl}_3\mathbb{C}$.
- *Objects:* generated by $\mathbb{1}, \circ, \circ\circ$.
- *Morphisms:* generated by



  and their vertical reflections. Taken up to isotopy as long as morphisms stay within the category.
- *Relations:* Same one-colour relations for both colours. New one-colour *wall* relations



Thank you for listening!