

The Diagrammatic Lusztig–Vogan Category for $SL(2, \mathbb{R})$

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Students Conference, 2024

Outline

1. Context and Motivation
2. Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$
3. Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$
4. Further work

Context and Motivation

Complex (simply-connected) Lie groups

Context and Motivation

Complex (simply-connected) Lie groups

BGG

Category \mathcal{O}

Vermas, Simple

Context and Motivation

Complex (simply-connected) Lie groups

BGG
Category \mathcal{O} \rightsquigarrow Hecke
algebra \mathcal{H}

Vermas, Simples

$\{\delta_s\}, \{b_s\}$

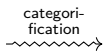
Context and Motivation

Complex (simply-connected) Lie groups

BGG
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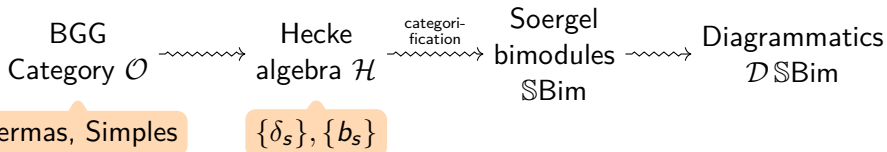
Soergel
bimodules
 $\mathbb{S}\text{Bim}$

Vermas, Simplex

$\{\delta_s\}, \{b_s\}$

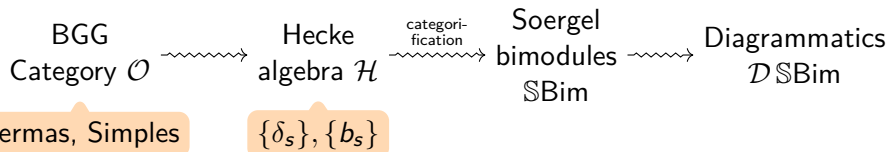
Context and Motivation

Complex (simply-connected) Lie groups



Context and Motivation

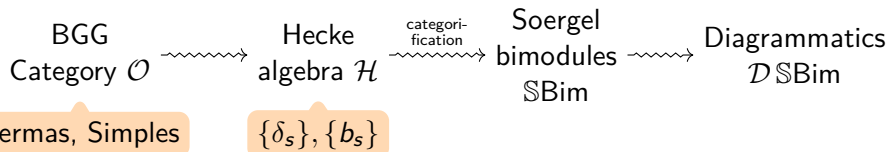
Complex (simply-connected) Lie groups



Real (reductive) Lie groups

Context and Motivation

Complex (simply-connected) Lie groups



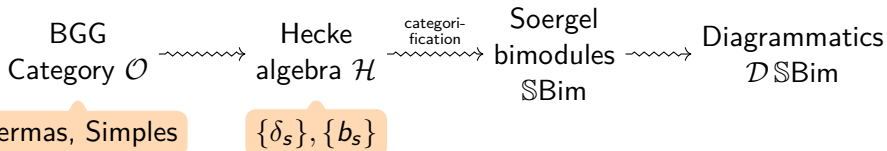
Real (reductive) Lie groups

Category of admissible reps.

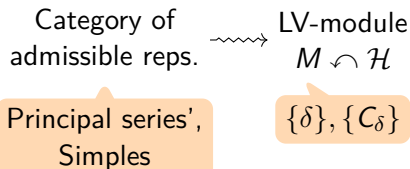
Principal series',
Simples

Context and Motivation

Complex (simply-connected) Lie groups

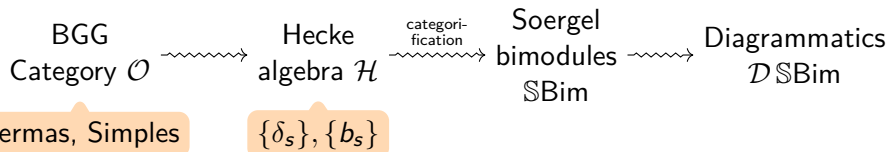


Real (reductive) Lie groups

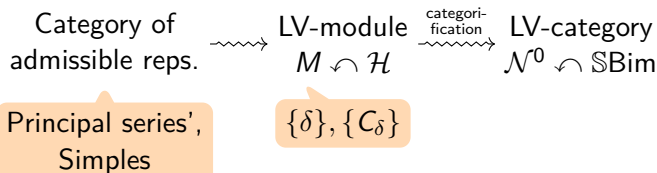


Context and Motivation

Complex (simply-connected) Lie groups

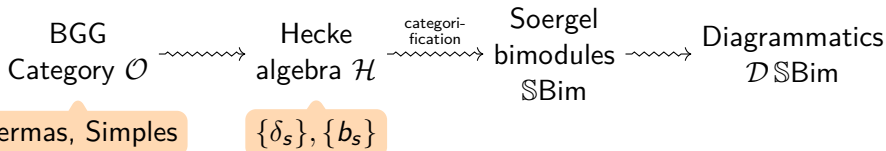


Real (reductive) Lie groups

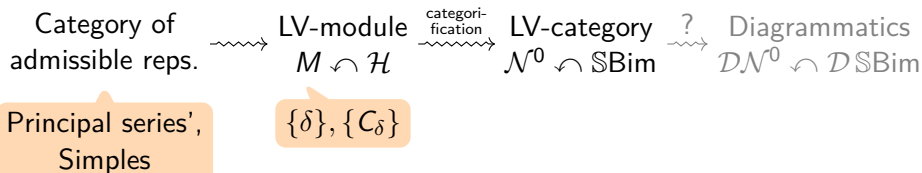


Context and Motivation

Complex (simply-connected) Lie groups



Real (reductive) Lie groups



Context and Motivation

Why diagrammatics?

- Diagrammatic methods enabled Elias & Williamson (2014) to give an algebraic proof of the Kazhdan–Lusztig conjecture (1979). This is more general than can be proved geometrically.
- Provided intuition for Williamson (2017) to discover counterexamples to Lusztig's conjecture (1980).
- $\mathcal{D}\mathbb{S}\text{Bim}$ can be studied in contexts where $\mathbb{S}\text{Bim}$ is not well-behaved, for example in fields of characteristic p .

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

The diagrammatic Hecke category \mathcal{DH} (for $\mathfrak{sl}_2\mathbb{C}$) is a $\mathbb{Z}[\frac{1}{2}]$ -linear monoidal category defined as follows.

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- *Objects*: generated by \bullet
i.e. $\mathbb{1}, \bullet, \bullet\bullet := \bullet \otimes \bullet, \bullet\bullet\bullet, \bullet\bullet\bullet\bullet, \dots$

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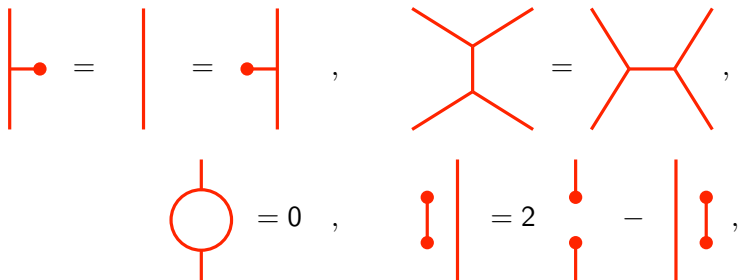


under relations...

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Definition

Relations:



and arbitrary planar isotopy.

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$$\frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \bullet \text{---} \bullet \\ | \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array}$$

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$$\begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \text{---} \quad \diagdown \\ | \quad \quad | \\ \diagdown \quad \text{---} \quad \diagup \end{array}$$

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Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

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$$\frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \text{---} \text{---} \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \text{---} \\ | \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \text{---} \\ | \\ \diagup \quad \diagdown \end{array} + \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \text{---} \text{---} \\ | \end{array}$$

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = 2 \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}$$

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Example

$$\frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \text{---} \text{---} \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \text{---} \\ | \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \text{---} \\ | \\ \diagup \quad \diagdown \end{array} + \frac{1}{2} \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \text{---} \\ | \\ \text{---} \text{---} \end{array}$$
$$= \begin{array}{c} | \quad | \\ \text{---} \text{---} \end{array}$$

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Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

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$$\begin{aligned} \frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ \text{---} \bullet \text{---} \bullet \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \bullet \text{---} \bullet \\ | \\ \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array} &= \frac{1}{2} \begin{array}{c} \diagup \quad \diagdown \\ | \\ \text{---} \bullet \text{---} \bullet \\ | \\ \diagdown \quad \diagup \end{array} + \frac{1}{2} \begin{array}{c} \text{---} \bullet \text{---} \bullet \\ | \\ \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array} \\ &= \begin{array}{c} | \quad | \\ \text{---} \bullet \quad \bullet \text{---} \\ | \quad | \end{array} = \begin{array}{c} | \quad | \end{array} \end{aligned}$$

Diagrammatic Soergel bimodules for $\mathfrak{sl}_2\mathbb{C}$

Theorem (Elias–Khovanov, 2010¹)

The Karoubi envelope of \mathcal{DH} is equivalent to the category of Soergel Bimodules $\mathbb{S}\text{Bim}$ for $\mathfrak{sl}_2(\mathbb{C})$ as graded additive \mathbb{R} -linear monoidal categories.

¹Ben Elias and Mikhail Khovanov. “Diagrammatics for Soergel categories”. In: *Int. J. Math. Math. Sci.* (2010), Art. ID 978635, 58.

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Definition

The diagrammatic LV-category \mathcal{DN}^0 (for $SL(2, \mathbb{R})$) is a $\mathbb{Z}[\frac{1}{2}]$ -linear right module category over \mathcal{DH} defined as follows. The right action by \mathcal{DH} is right concatenation.

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- *Objects*: generated by $\mathbb{1}$ and \circ
i.e. $\mathbb{1}$, \bullet , $\bullet\bullet$, $\bullet\bullet\bullet$, $\bullet\bullet\bullet\bullet$, ... and \circ , $\circ\circ$, $\circ\circ\circ$, ...

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- *Morphisms*: generated by

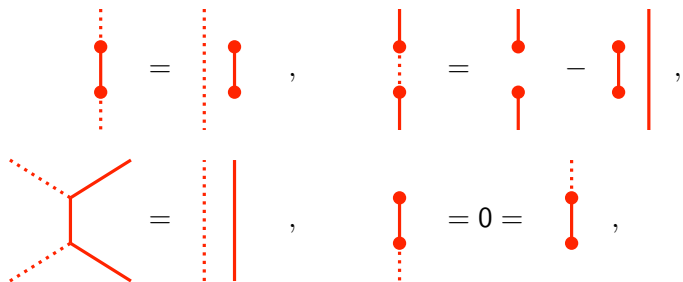


under relations...

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Definition

Relations:



and arbitrary planar isotopy while dotted red lines never appear right of any red.

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Definition

Remark

This diagrammatic definition is not entirely new: Elias–Williamson² had constructed very similar diagrammatics for localisation of Soergel bimodules. The LV-category is a subcategory of this, with restrained objects, morphisms and relations.

²Ben Elias and Geordie Williamson. “Soergel calculus”. In: *Represent. Theory* 20 (2016), pp. 295–374.

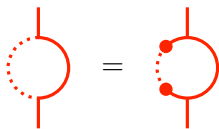
Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Example



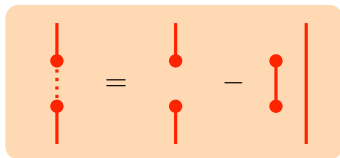
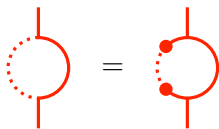
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Example



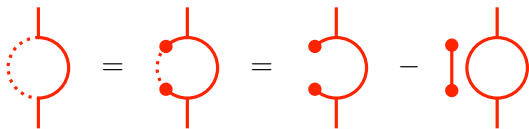
Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Example



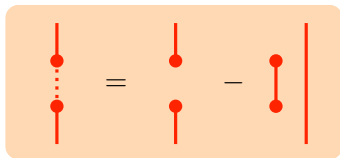
Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Example



A diagrammatic equation involving four terms. The first term is a circle with a dashed arc on its left side and two vertical lines extending from the top and bottom. The second term is a circle with a solid arc on its left side, two red dots on the arc, and two vertical lines. The third term is a circle with a solid arc on its left side, two red dots on the arc, and two vertical lines. The fourth term is a vertical line with two red dots and a circle with two vertical lines. The terms are connected by equals and minus signs.

$$\text{Circle with dashed arc} = \text{Circle with solid arc and dots} = \text{Circle with solid arc and dots} - \text{Vertical line with dots and circle}$$



A diagrammatic equation inside an orange rounded rectangle. The first term is a vertical line with two red dots, the upper segment being dashed. The second term is a vertical line with two red dots. The third term is a vertical line with two red dots. The terms are connected by equals and minus signs.

$$\text{Vertical line with dashed segment and dots} = \text{Vertical line with dots} - \text{Vertical line with dots}$$

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Example

The diagram shows an equality between four terms. The first term is a circle with a vertical line extending upwards and another extending downwards. The left half of the circle is drawn with a dotted line. This is equal to the second term, which is a solid circle with two red dots on its left side, one near the top and one near the bottom. This is equal to the third term, which is a solid circle with two red dots on its left side, one near the top and one near the bottom, but the left side of the circle is open. This is equal to the fourth term, which is a solid circle with a vertical line extending upwards and another extending downwards, and a vertical line segment to its left with two red dots at its ends.

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Example

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} - \text{Diagram 4} = \text{Diagram 5}$$

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Theorem (Z.)

The Karoubi envelope of \mathcal{DN}^0 is equivalent to the Lusztig–Vogan category \mathcal{N}^0 for $SL(2, \mathbb{R})$ as graded additive \mathbb{R} -linear right module categories over $\mathbb{S}\text{Bim}$ (for $\mathfrak{sl}_2\mathbb{C}$).

Diagrammatic Lusztig–Vogan category for $SL(2, \mathbb{R})$

Proof idea

The backbone of the proof is the isomorphism $\bullet \simeq \circ\bullet$, given by the relations

The diagram shows two equations. The first equation shows a vertex with two solid lines extending downwards and two dotted lines extending upwards, equal to two vertical lines, one solid and one dotted. The second equation shows a circle with a solid line on the left and a dotted line on the right, equal to a single solid vertical line.

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} .$$

This reflects an isomorphism on the level of modules that allows us to construct a basis of morphisms from an existing one in \mathcal{DH} .

Further Work

- There is an isomorphism $SL(2, \mathbb{R}) \simeq SU(1, 1)$, so a natural next step is to consider $SU(2, 1)$. The aim is to define diagrammatics for the infinite family $SU(n, 1)$.
- Understand the dimension of the general morphism spaces at the level of the Lusztig–Vogan modules.

Extras

A Glimpse of $SU(2,1)$

\mathcal{DH} for $\mathfrak{sl}_3\mathbb{C}$

- *Objects:* generated by $\mathbb{1}, \bullet, \bullet$.
- *Morphisms:* generated by



and their vertical reflections. Taken up to arbitrary isotopy.

- *Relations:* Same one-colour relations. New two-colour relations (omitted).

Extras

A Glimpse of $SU(2, 1)$

\mathcal{DN}^0 for $SU(2, 1)$

- Right module category over \mathcal{DH} for $\mathfrak{sl}_3\mathbb{C}$.

Extras

A Glimpse of $SU(2, 1)$

\mathcal{DN}^0 for $SU(2, 1)$

- Right module category over \mathcal{DH} for $\mathfrak{sl}_3\mathbb{C}$.
- *Objects*: generated by $\mathbb{1}$, \circ , $\circ\circ$.

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A Glimpse of $SU(2,1)$

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- Right module category over \mathcal{DH} for $\mathfrak{sl}_3\mathbb{C}$.
- *Objects*: generated by $\mathbb{1}, \circ, \circ\circ$.
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and their vertical reflections. Taken up to isotopy as long as morphisms stay within the category.

Extras

A Glimpse of $SU(2,1)$

\mathcal{DN}^0 for $SU(2,1)$

- Right module category over \mathcal{DH} for $\mathfrak{sl}_3\mathbb{C}$.
- *Objects*: generated by $\mathbb{1}, \circ, \circ\circ$.
- *Morphisms*: generated by



and their vertical reflections. Taken up to isotopy as long as morphisms stay within the category.

- *Relations*: Same one-colour relations for both colours. New one-colour *wall* relations

$$\begin{array}{|c} \bullet \\ \hline \end{array} = \begin{array}{|c} \hline \end{array}, \quad \begin{array}{|c} \hline \end{array} = \begin{array}{|c} \hline \end{array}, \quad \begin{array}{|c} \hline \end{array} = - \begin{array}{|c} \hline \end{array}.$$

Thank you for listening!