Graph Colouring, SO(3) Webs, and the B1 knot invariant

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UNSW Postgrad Seminar

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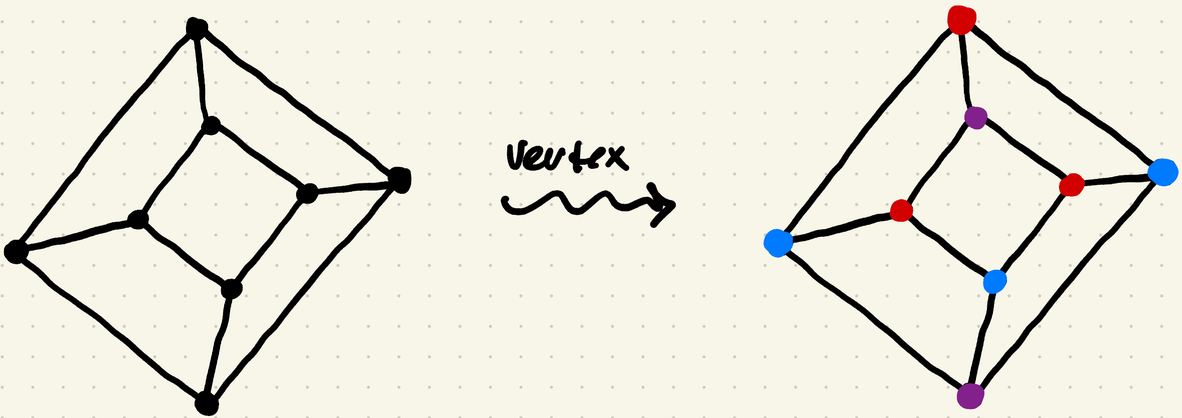
Outline

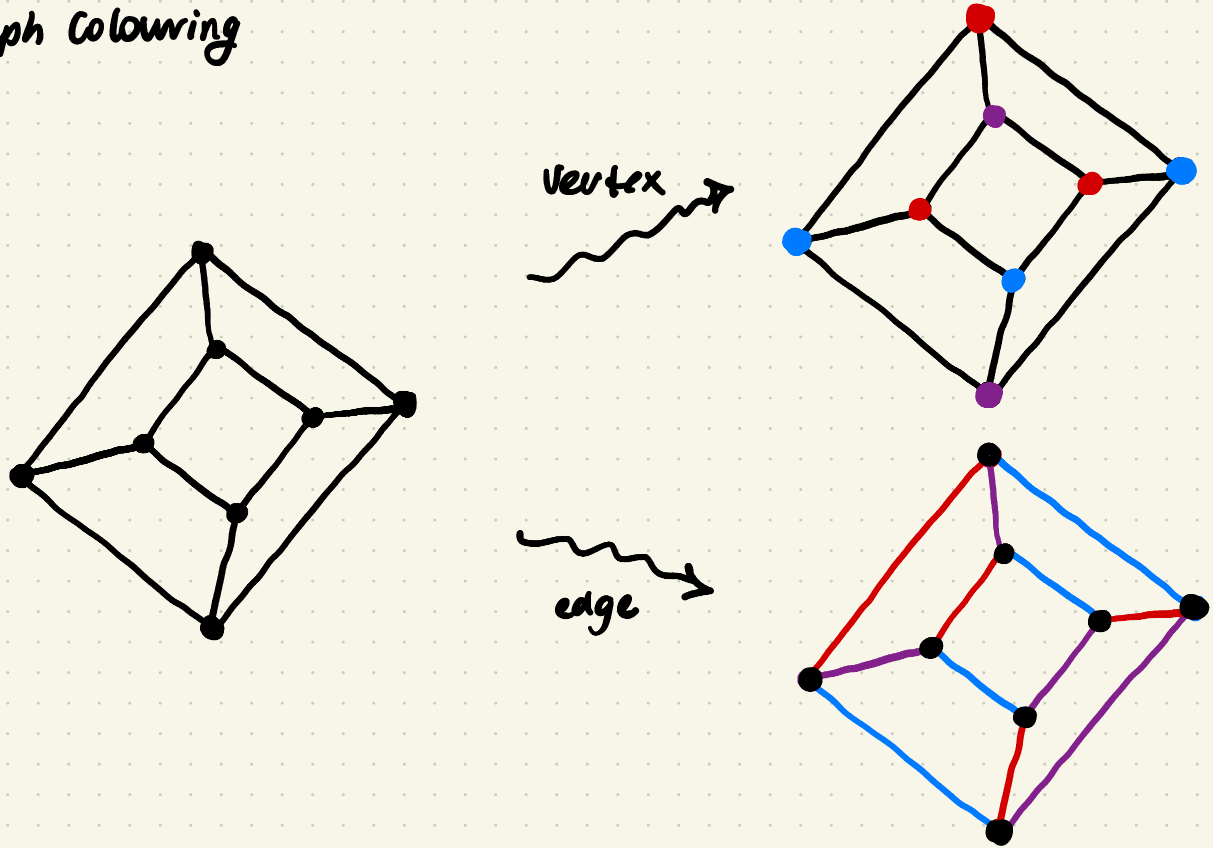
1. Graph colouring
2. Category of SO(3) webs
   - Aside: 4-colour theorem
   - Aside: where is SO(3)?
3. The B1 knot invariant

1. Graph Colouring



adjacent vertices have different colours

1. Graph Colouring



adjacent vertices have different colours

adjacent edges have different colours

Basic Question

Given a graph and $n \in \mathbb{Z}_{>0}$, can it be vertex (or edge) coloured with n colours?

2. Category of SO(3) webs

Overview of diagram categories

"Category" = { objects, some morphisms w/ id & composition }

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"monoidal category" = "category" where we have an "associative" "tensor product" $\otimes$ for objects and morphisms, with a "unit" object $\mathbb{1}$

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Overview of diagram categories

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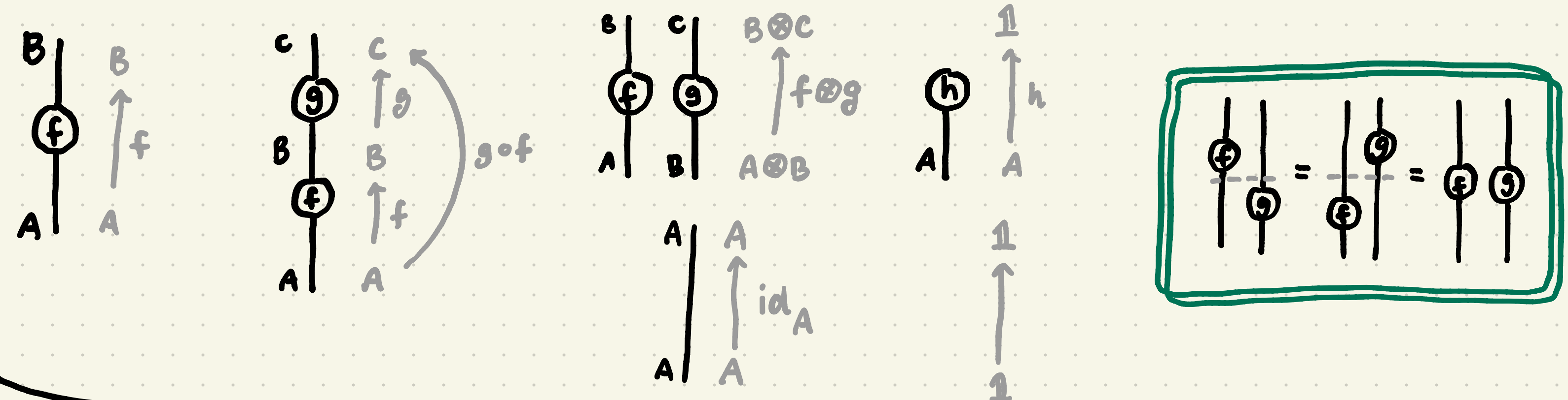
Can be drawn



2. Category of SO(3) webs

<u>DEFINITION</u> ($\mathbf{Web}(SO(3))$) "$\mathbb{Z}$-linear", "Pivotal category"

Objects: generated by $\bullet$

i.e. $\mathbb{1}$, $\bullet$, $\bullet\bullet = \bullet \otimes \bullet$, $\bullet\bullet\bullet$, etc.

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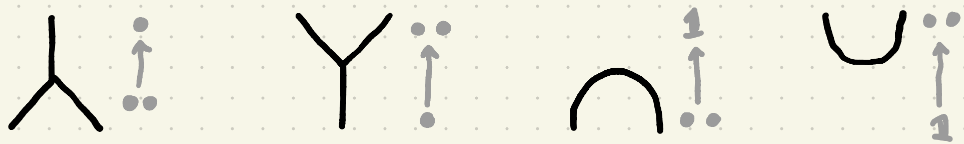
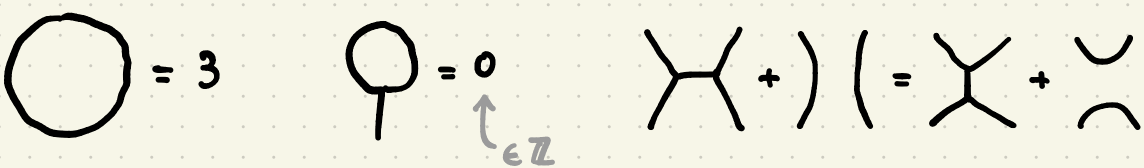
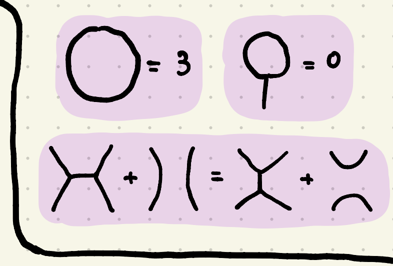
Morphisms: generated by



modulo planar isotopy and local relations

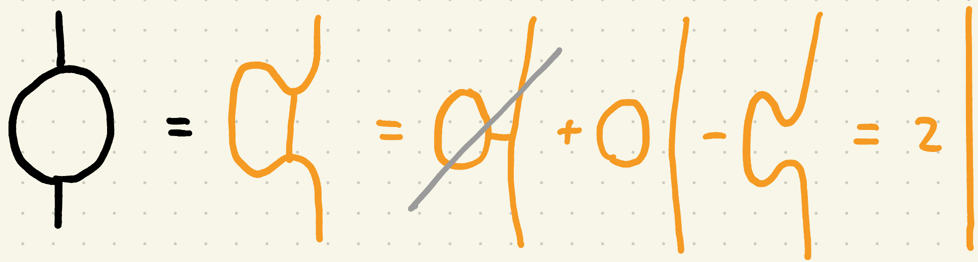


2. Category of SO(3) webs



Example

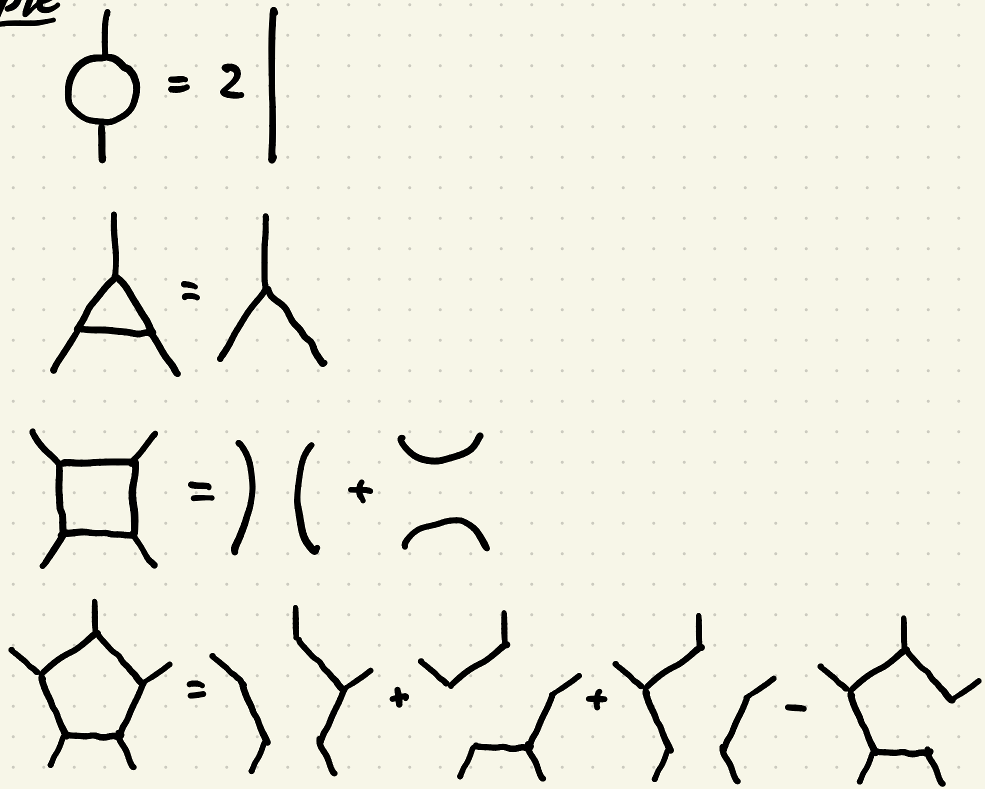
①



②

2. Category of SO(3) webs

Example



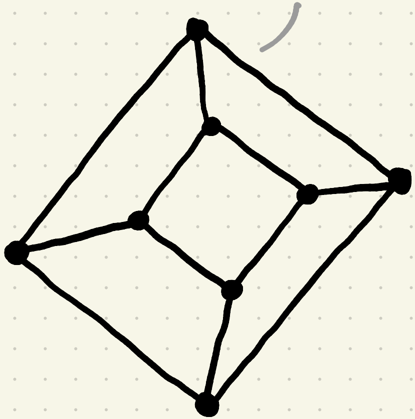
etc. etc.

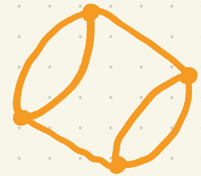
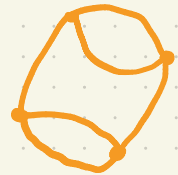
2. Category of SO(3) webs

Q Where is the "graph colouring" that you promised?

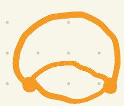
Example

remember me?



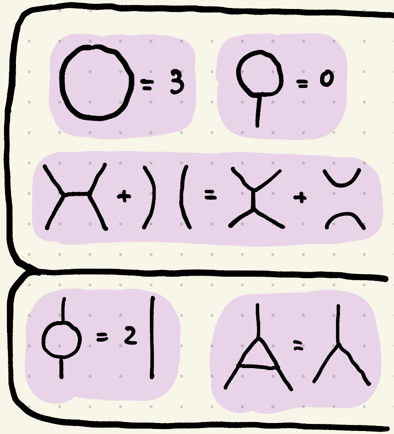
=  + 

= 2 

= 4 

= 8 

= 24



2. Category of $SO(3)$ webs

PROPOSITION

- $\mathrm{Hom}_{\mathrm{Web}(SO(3))}(\mathbb{1},\mathbb{1}) = \mathbb{Z}\langle \mathrm{id}_{\mathbb{1}} \rangle$
- The "evaluation" of a 3-regular graph in $\mathrm{Web}(SO(3))$ is the number of 3-edge colourings of the graph

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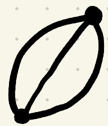
Simpler example:  How many 3-edge colourings?

$3! = 6$ (assign a different one to each edge)

Using the above proposition:

 $= 2$  $= 6$

2. Category of SO(3) webs

PROPOSITION

- $\mathrm{Hom}_{Web(SO(3))}(\mathbb{1},\mathbb{1}) = \mathbb{Z}\langle \mathrm{id}_{\mathbb{1}} \rangle$
- The "evaluation" of a 3-regular graph in $Web(SO(3))$ is the number of 3-edge colourings of the graph

We can define a "quantum" variant of $Web(SO(3))$ by taking $\mathbb{Z}[q^{\pm 1}]$ and replacing

$$\bigcirc = [3]_q := q^2 + 1 + q^{-2} = \overset{q^2+2+q^{-2}}{n} - 1.$$

2. Category of SO(3) webs

PROPOSITION

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q^2+2+q^{-2}

$$\bigcirc = [3]_q := q^2+1+q^{-2} = n-1.$$

PROPOSITION

- $\mathrm{Hom}_{\mathcal{Web}(SO(3))_q}(\mathbb{1},\mathbb{1}) = \mathbb{Z}[q^{\pm 1}]\langle id_{\mathbb{1}} \rangle$
- The "evaluation" of a 3-regular graph in $\mathcal{Web}(SO(3)_q)$, written in terms of n, is it's chromatic polynomial.

2. Category of SO(3) webs

Aside: 4-colour theorem

THEOREM (Guthrie 1850's conjectured; Appel-Haken 1976 first proved)

Any loopless planar graph can be 4-vertex coloured

THEOREM (Tait 1870's)

The 4-colour theorem is equivalent to

"every planar bridgeless 3-regular graph can be 3-edge colourable"

2. Category of SO(3) webs

Aside: Where is SO(3)?

By an observation of Yamada & Turaev ~1989:

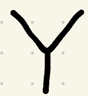
$$\mathrm{Web}(SO(3))_{\mathbb{C}} \xrightarrow{\text{faithful monoidal}} \mathrm{Rep}\, SO(3) \quad \text{finite dim. over } \mathbb{C}$$

objects

$\bullet \longmapsto \mathbb{C}^3$ natural representation given by matrix multiplication

morphisms

$\cup, \cap \longmapsto$ unit & counit of $\mathbb{C}^3$ as a self dual object

 $\longmapsto$ inclusion $\mathbb{C}^3 \hookrightarrow \mathbb{C}^3 \otimes \mathbb{C}^3$

 $\longmapsto$ (use functoriality & monoidality)

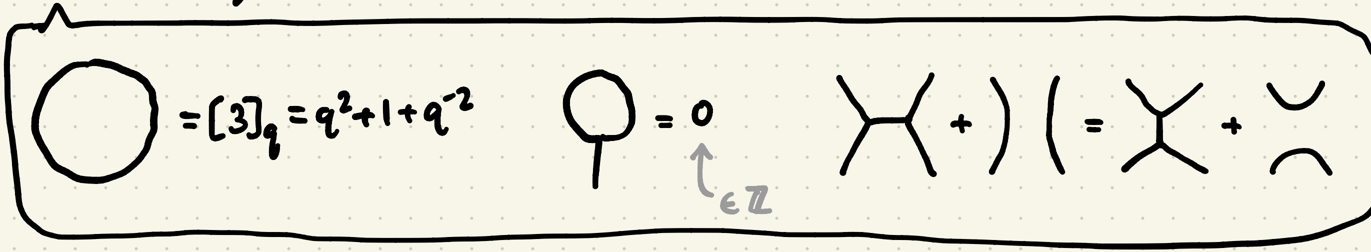
This functor is an equivalence of categories up to some abelianisation

PROPOSITION

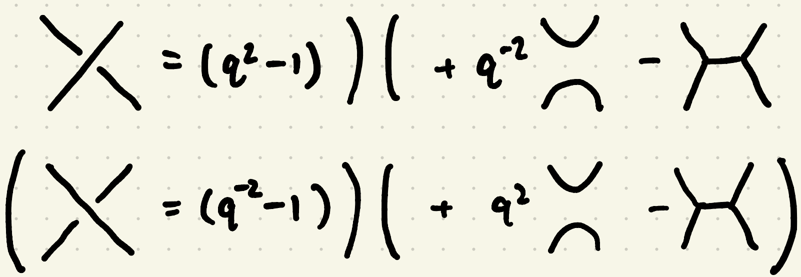
Let G be compact Lie group, and V a faithful rep. of G. Then every irreducible rep. of G is a subrepresentation of $V^{\otimes n} (V^*)^{\otimes m}$ for some $n, m \in \mathbb{Z}_{\geq 0}$.

3. The B1 knot invariant

Take Web($SO(3)_q$)

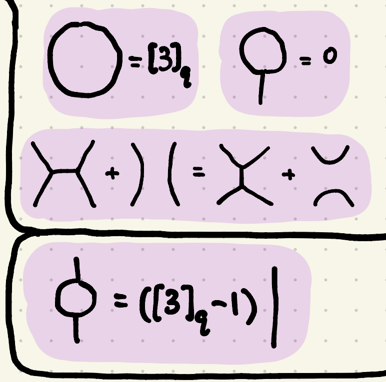


We have a crossing



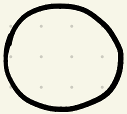
(this makes Web($SO(3)$) a braided category)

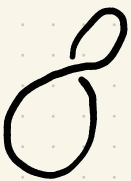
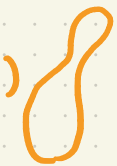
<u>Observation</u>: We can write knots as a $\mathbb{Z}[q^{\pm 1}]$-linear combination of 3-regular "graphs"!

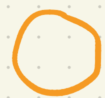


3. The B1 knot invariant

Example

 $= q^2 + 1 + q^{-2} = [3]_q$

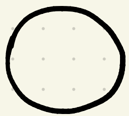
 $= (q^2 - 1)$  $+ q^{-2}$  $-$ 

$= (q^2 \cancel{-1} + q^{-2}[3]_q - ([3]_q \cancel{-1}))$ 

$= q^4 [3]_q$

3. The B1 knot invariant

<u>Example</u>

 $= q^2 + 1 + q^{-2} = [3]_q$

 $= q^4 [3]_q = q^6 + q^4 + q^2$

 $= q^{12} - q^8 - q^6 - q^4 + q^{-2} + q^{-4} + q^{-6} + q^{-10}$

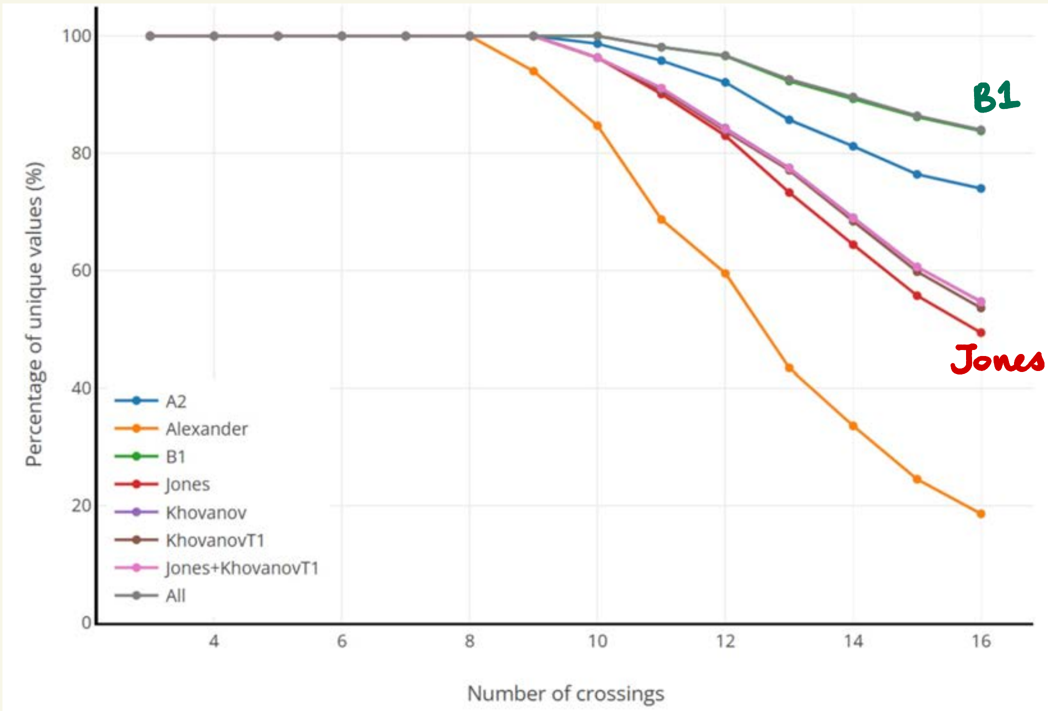
3. The B1 knot invariant

How does it compare with the Jones invariant?

(Joint w/ Daniel Tubbenhauer, 2025)

Interactive things

under "stats"



% unique values
(higher is better)

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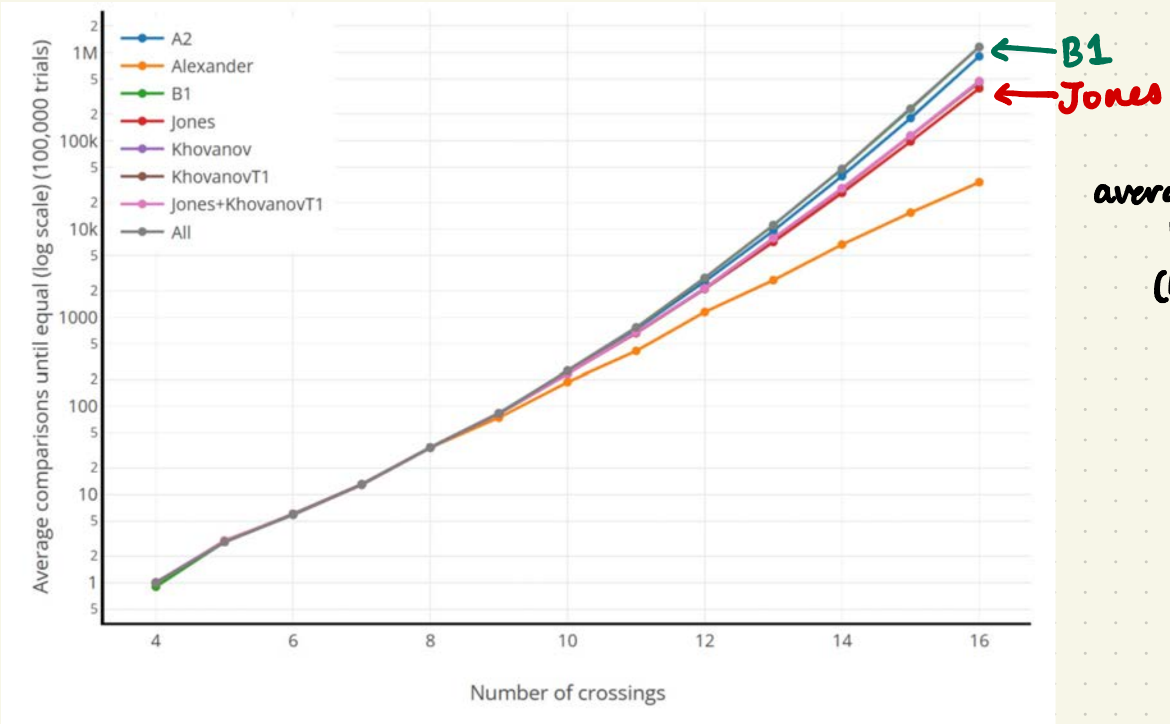

average no. comparisons until equal

(higher is better)

3. The B1 knot invariant

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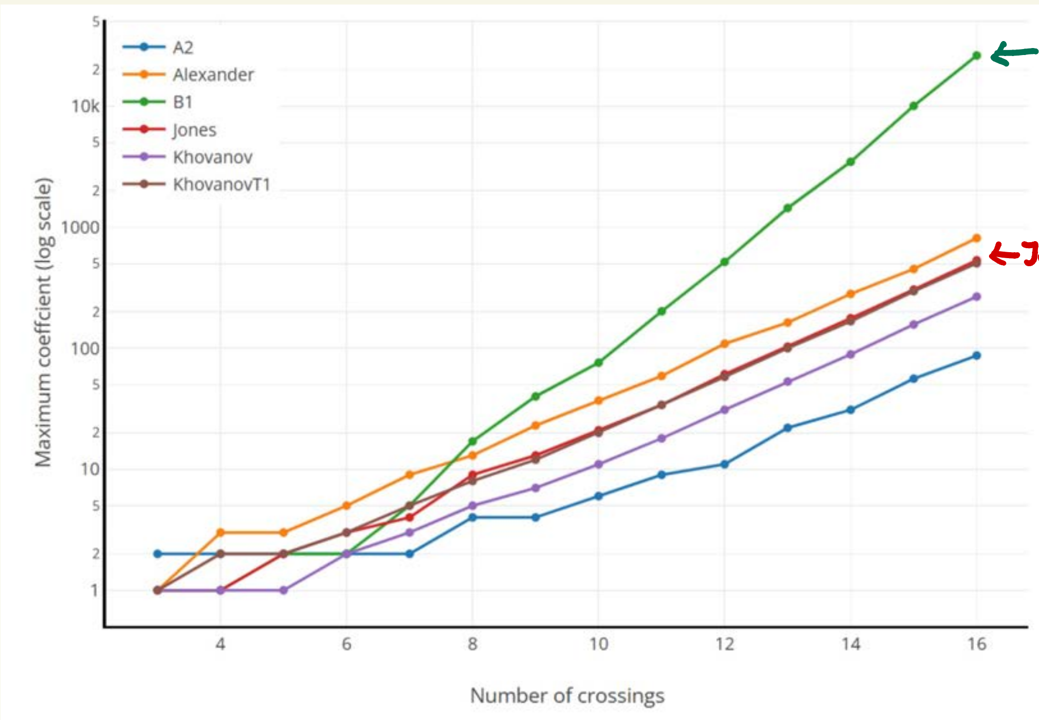

← B1

← Jones

maximal coeff

(lower is better)

Slides are somewhere here...

The End!