Bruhat orders: classical and for Lusztig-Vogan

Outline
- Classical Bruhat Order
  - On the Weyl group
  - $B\backslash G/B$

- LV's Bruhat order
  - $K\backslash G/B$ order
  - several definitions for the module

- Atlas

① Classical Bruhat order
First, combinatorially.
- Fix a Coxeter system (W, S). For all our examples we take $W = S_3 = \langle s,t \mid s^2 = 1,\ sts = tst \rangle$
  $S = \{s,t\}$

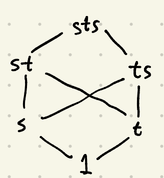
- An <u>expression</u> of $w \in W$ is a tuple $(s_1, \dots, s_k) \in \coprod_{n\geq 0} S^n$
  such that $w = s_1 s_2 \dots s_k$.
  eg. (s,t,s,s,t) is an expression of $stsst = stt$
  $= s$

  We say an expression of w is <u>reduced</u> if it has minimal length
- <u>Rk</u>: Reduced expressions may <u>not</u> be unique
  eg. (s,t,s) and (t,s,t) are reduced expressions of $sts = tst \in W$.

  THM (Matsumoto 1964)
  Any two reduced expressions of $w \in W$ are related by braid relations.
- The <u>length</u> of $w \in W$, denoted $\ell(w)$, is the length of its reduced expression.

DEF The (strong) <u>Bruhat order</u> on W is a partial order $\leq$ defined as follows.
For $u, v \in W$, say $u \leq v$ iff some subexpression of a reduced expression of v is a (reduced) expression of u.

eg. $sts \in W$ has reduced expr. (t,s,t)
which has subexpressions $\emptyset, (t), (s), (t,s), (s,t), (t,s,t)$
eg. $st \in W$ has reduced expr. (s,t)
which has subexpressions $\emptyset, (s), (t), (s,t)$
The Hasse diagram of the Bruhat order on S_3 is



Why do we care?
- The Bruhat order controls terms appearing in KL-basis of Hecke alg. $\mathcal{H}(W,S)$:

$$b_x = \delta_x + \sum_{y<x} h_{y,x}\, \delta_y$$

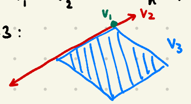
and these are related to KL-cells and decomposition numbers.

On B-orbits of G/B

- We do an example to illustrate the idea
- Let $G = SL_3(\mathbb{C}) = \{ M \in M_{3\times 3}(\mathbb{C}) : \det M = 1 \}$

  Take standard Borel subgroup
  $$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \subset SL_3(\mathbb{C})$$
  Take standard maximal torus
  $$T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \subset B \subset SL_3(\mathbb{C})$$
- The full flag variety is $X := G/B$
  - A full flag of vector space V is a chain of inclusions of subspaces
    $\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ where $\dim V_i = i$

    eg. $n=3$:



looks like a flag

  - G/B has points given by full flags in $\mathbb{C}^3$.

Q How do we see this?

- Fix a basis for $V = \mathbb{C}^3$. Let any 3×3 invertible matrix $g \in GL_3(\mathbb{C})$ act on a complete flag
  $$\{0\} \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3$$
  by sending it to another full flag.
  $$\{0\} \subset gV_1 \subset gV_2 \subset gV_3 = \mathbb{C}^3 .$$
  Think of g as "changing the basis". For example: the standard flag
  $$\{0\} \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle = \mathbb{C}^3 \quad \text{where } \{e_i\} \text{ are the standard basis,}$$
  is sent to
  $$\{0\} \subset \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rangle \subset \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix} \right\rangle \subset \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle = \mathbb{C}^3$$
  by the action of
  $$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix} \in GL_3(\mathbb{C}).$$
- It is clear that $GL_3(\mathbb{C})$ acts transitively on the set of flags in $\mathbb{C}^3$
  $$\underset{\text{any flag}}{0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, v_3 \rangle} \xrightarrow{(v_1\ v_2\ v_3)^{-1}} \underset{\text{std. flag}}{0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle} \xrightarrow{(u_1\ u_2\ u_3)} \underset{\text{any other flag}}{0 \subset \langle u_1 \rangle \subset \langle u_1, u_2 \rangle \subset \langle u_1, u_2, u_3 \rangle}$$
  You can also think about any matrix in $GL_3(\mathbb{C})$ as a flag. This is not unique: $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$ represent the same flag.
- We have $\mathrm{Stab}_{GL_3(\mathbb{C})}(0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle) = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \subset GL_3(\mathbb{C})$

  "$\supseteq$" is clear

  "$\subseteq$" because if $g \in GL_3(\mathbb{C})$ and $g\langle e_1, \dots, e_i \rangle \subseteq \langle e_1, \dots, e_i \rangle$,

  then $ge_i = \alpha_1 e_1 + \dots + \alpha_i e_i$ because $\langle g e_i \rangle \subseteq g\langle e_1, \dots, e_i \rangle \subseteq \langle e_1, \dots, e_i \rangle$

  so g must be upper triangular.
- Then $GL_3(\mathbb{C}) / \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$ = set of flags up to iso. of vector spaces

  This is because $g_1, g_2 \in GL_3(\mathbb{C})$ define the same flag if $g_2^{-1} g_1$ stabilises std. flag, ie. if $g_2^{-1} g_1 \in \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \subset GL_3(\mathbb{C})$

$$\begin{tikzcd} \text{std. flag} \arrow[r, "g_1"] \arrow[r, shift right=3, "g_2"'] & \text{flag}_1 \\ & \text{flag}_2 \end{tikzcd} \qquad \begin{tikzcd} \text{std. flag} \arrow[r, bend left, "g_1"] \arrow[d, equal, "?"'] & \text{flag}_1 \arrow[dl, bend left, "g_2^{-1}"] \\ ?? & \end{tikzcd}$$

- If you were watching closely, you may notice that aI for $a \in \mathbb{C}$ act trivially on flags (just scale the bases).
  So we may rescale all the matrices to have $\det = 1$, ie look at $SL_3(\mathbb{C})$ instead. The whole story is the same

- B acts on G/B on the left by matrix multiplication, ie. by usual matrix action on flags.
  What are the B orbits? ie. what is $B\backslash G/B$?

THM (Bruhat Decomposition)
Let G be conn. reductive alg. group over alg. closed field, $B \subseteq G$ a Borel subgroup, W a Weyl group of G corresp. to a maximal torus in B. Then

$$G = \coprod_{w \in W} BwB.$$

"Bruhat cells"

eg. $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

Remark
In $GL_n(\mathbb{C})$, the Bruhat decomposition is effectively "Gaussian elimination" because we have $W = S_n = \{\text{permutation matrices in } GL_n(\mathbb{C})\}$, so

$$g = b_1 w b_2 \iff w^{-1} b_1^{-1} g = b_2$$

should be read as
"Take invertible matrix g,
add and multiply rows with b_1^{-1} (adding lower rows to higher),
then permute the rows with w^{-1}.
The result is an invertible upper triangular matrix".

Hence $B\backslash SL_3(\mathbb{C})/B \simeq W = S_3$.

- This decomposition gives a stratification

$$G/B = \coprod_{w \in W} BwB/B$$

these are $\simeq \mathbb{A}^{\ell(w)}$, closures contain the information

into *schubert cells* $C_w := BwB/B$, which are the B orbits in G/B.
Schubert varieties are the topological closures $\overline{C_w}$ inside G/B.

DEF The Bruhat order on the Weyl group of G is given by
$$v \leq w \text{ iff } \overline{C_v} \subseteq \overline{C_w} \quad (\text{or equiv. } C_v \subseteq \overline{C_w})$$
where $v, w \in W$.

THM The two definitions of "Bruhat order" agree.

Some facts

1. Let Δ_w be Verma module corresp. to weight $w \cdot 0$ in category $\mathcal{O}$ (principal block)
   Then $v \leq w$ iff $\Delta_w \subseteq \Delta_v$.

2. Let Δ_w be as above and L_w be the unique simple quotient of Δ_w.
   Then $[\Delta_v : L_w] \neq 0$ iff $v < w$ (Verma, BGG, van den Hombergh)

- KL-conjecture says $[\Delta_v : L_w] = P_{v,w}(1)$, evaluation of KL-poly. (Beilinson-Bernstein, Brylinski-Kashiwara)

② Bruhat Order on the LV-module

For K orbits on G/B

- We can work out $K\backslash G/B$ by a similar example
  - $G_{\mathbb{C}} = SL_3(\mathbb{C})$, $\sigma : g \longmapsto \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix} {}^t\bar{g}^{-1} \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}$ antiholomorphic involution

    $\theta : g \longmapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} g \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ holomorphic involution

    $G_{\mathbb{R}} = G_{\mathbb{C}}^{\sigma} = SU(2,1)$

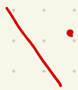
    $K = G_{\mathbb{C}}^{\theta} = \left(\begin{array}{c|c} M & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 \; 0 & \det M^{-1} \end{array}\right)$ where $M \in GL_2(\mathbb{C})$
  - Picture

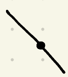
$$\begin{tikzcd} & G_{\mathbb{C}} = SL_3(\mathbb{C}) \arrow[dl, "\theta"'] \arrow[dr, "\sigma"] & \\ K = \left(\begin{array}{c|c} M & \\ \hline & \det M^{-1} \end{array}\right) & & G_{\mathbb{R}} = G_{\mathbb{C}}^{\sigma} = SU(2,1) \end{tikzcd}$$

  - We get the same flag variety $X = G_{\mathbb{C}}/B$ where $B \subset SL_3(\mathbb{C})$ is standard Borel

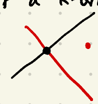
- We can calculate $K\backslash G/B$ explicitly
  - See $G/B = \{\text{flags } 0 \subsetneq V_1 \subsetneq V_2 \subseteq \mathbb{C}^3\} = \{\text{line} \subset \text{plane in } \mathbb{C}^3\} = \{\text{point} \subset \text{line in } \mathbb{CP}^2\}$
  - Notice that K stabilises the $\langle e_1, e_2 \rangle$ plane and the $\langle e_3 \rangle$ line.

    In $\mathbb{CP}^2$, we draw this as (following Anna's notes)



Flags in G/B we draw as 

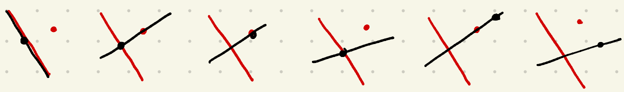
An example of a K-orbit is



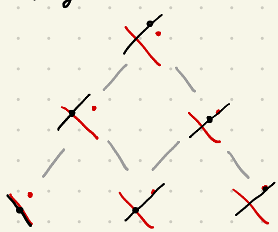
of flags whose 1-dimensional part lie on the $\langle e_1, e_2 \rangle$-plane and whose 2-dimensional part does not.

Q why is K transitive on this set ? Also on the below sets ?

We have these orbits altogether



which, taking closures has this inclusion order



However this is not enough for the module

Order on the LV-module

- The module has a basis indexed by

$\mathcal{D} = \{ (\mathcal{O}, \mathcal{L}) \quad : \quad \mathcal{O}$ is a K-orbit of G/B and $\mathcal{L}$ is a K-equivariant local system on $\mathcal{O} \}$

ie. there is more information than just $K \backslash G/B$

- The Bruhat order(s) here is an imitation of the Bruhat order in the $B\backslash G/B$ case

DEF (Lusztig-Vogan 1983)

The Bruhat $G_{\mathbb{R}}$-order on $\mathcal{D}$ is smallest order such that the following holds.
Let $\delta, \delta' \in \mathcal{D}$ where δ appears in $H_s \delta'$ w/ non-zero coeff and $\ell(\delta) = \ell(\delta') + 1$. $\ell((\mathcal{O}, \mathcal{L})) = \dim \mathcal{O}$
Let $\gamma, \gamma' \in \mathcal{D}$ where γ appears in $H_s \gamma'$ w/ non-zero coeff and $\ell(\gamma) = \ell(\gamma') + 1$.
If $\gamma' \leq \delta'$ then $\gamma \leq \delta$ and $\delta' \leq \delta$.

- One basic case is if $\gamma = \delta$ and $\gamma' = \delta'$ then we immediately get $\delta' \leq \delta$.
- The paper states this reduces to the $B\backslash G/B$ case.

Another Bruhat order on a block $\mathcal{B}$ of regular characters for G w/ infinitesimal character χ.
unlike last week where $\mathcal{B}$ = blocks of irred. HC-modules

DEF (Vogan (IV), 1982 ; Def 12.9)

The Bruhat order on $\mathcal{B}$ is the smallest ordering such that

$$\phi \leq \gamma \quad \text{if} \quad m(\bar{\pi}(\phi), \pi(\gamma)) \neq 0$$

where $\pi(\phi)$ is std. rep. corresp to ϕ,
$\bar{\pi}(\phi)$ is irred. corresp. to std. rep $\pi(\phi)$
and $m(\bar{\pi}(\phi), \pi(\gamma))$ is composition multiplicity of $\bar{\pi}(\phi)$ in $\pi(\gamma)$.

Vogan : this is a theorem for usual Bruhat order and Vermas, taken as def.

In the same paper, Vogan defines the Bruhat r-order :
the smallest order containing the Bruhat order and constant on each $\mathcal{G}^r(\gamma)$
where $\mathcal{G}^r(\gamma)$ is smallest subset of $\mathcal{B}$ containing γ and closed under cross action of simple <u>real</u> reflections $s \in S(\gamma)$

Also the Bruhat c-order w/ $\mathcal{G}^c(\gamma)$ and simple complex roots.

These last two are related to

DEF (Vogan (IV), 1982 ; Def 12.7)

Let $\phi, \phi' \in \mathcal{B}$ and $s \in \mathcal{B}^a$, we say $\phi \xrightarrow{s} \phi'$ iff
(a) $\ell^I(\phi') = \ell^I(\phi) - 1$
(b) ϕ' appears in $T_s \phi$

(see lemma 12.13, 12.18).

③ Comment on Atlas's order on $K\backslash G/B$

- Let P_α be parabolic subgroup of G corresp to root α.
- We have a natural projection

$$\pi_\alpha : G/B \longrightarrow G/P_\alpha$$

and can define

$$\pi^\alpha := \pi_\alpha^{-1} \circ \pi_\alpha$$

that sends B-cosets to P_α-cosets modulo B.

- Atlas' $K\backslash G/B$ graph is given by
  - vertices $= K\backslash G/B$, writing Q_v for G/B orbit of vertex v.
  - edges: $v_1 \longrightarrow v_2$ iff $\dim(Q_{v_1}) = \dim(Q_{v_2}) + 1$ and $Q_{v_1} \subset \pi^\alpha(Q_{v_2})$ for some simple root α