Grassmannians and Scubert cells (2025/07/31)

Outline

1. Motivations
2. Grassmannians
3. Scubert cells & schubert varieties
4. Plücker coordinates and Plücker relations

Resources (what the content is based off)

- (Gr.) The Grassmannian as a Projective variety (Hudec)
- (Gr.) Algebraic geometry, A first course (Harris)
- (Gr.) Basic Algebraic Geometry I (Shafarevich)
- (Sch.) Variations on a theme of Schubert Calculus (Gillespie)

① Motivation

1. Counting points of intersection (Schubert calculus)
   - Hermann Schubert (19th century) solved various enumeration problems in projective geometry
   - Hilbert's 15th problem : construct rigorous foundation to Schubert's techniques
2. Generalising projective space
   - $\mathbb{P}^{n-1}$ is the space of lines in $\mathbb{C}^n$
   - Q can we be more general and consider k-dimensional subspaces ?

② Basics of Grassmannians

DEF The Grassmannian $Gr(k,V)$, for $\mathbb{C}$-vector space V and $k \le \dim V$, is the space of k-subspaces of V.

- We will see later this is a projective variety
- Concretely, we can look at $Gr(k,\mathbb{C}^n)$.
  - each point is a k-subspace with basis $\{v_1,\dots,v_k\}$, which can be expressed by a $k\times n$ matrix

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

eg. in $Gr(3,\mathbb{C}^7)$ we have

$$\begin{pmatrix} 0 & -1 & -3 & -1 & 6 & -4 & 5 \\ 0 & 1 & 3 & 2 & -7 & 6 & -5 \\ 0 & 0 & 0 & 2 & -2 & 4 & -2 \end{pmatrix}$$

- This is too fine: two bases for same subspace have different matrices
- **FACT** Every matrix has a unique reduced row echelon form

  COR Matrices A and B (w/ same shape) have the same row space iff they have the same reduced row echelon form

  └ proof idea: $A = MB$ for M row operations

  So, we can represent each point in $Gr(k,\mathbb{C}^n)$ by a full rank matrix in rref.

  eg (above) has rref $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 & -5 & 2 & 0 \end{pmatrix}$

Some facts

- there is a bijection $Gr(k,\mathbb{C}^n) \longleftrightarrow \{n\times k \text{ matrices of rk } k\} \big/ GL_k(\mathbb{C})$

  └ via left multiplication
- $\dim Gr(k,\mathbb{C}^n) = k(n-k)$
  - since $\dim(n\times k \text{ matrices of rk } k) = nk$ and $\dim GL_k(\mathbb{C}) = k^2$
- $Gr(k,V) \cong GL(V) \big/ Stab(V_0)$ where $V_0 \subseteq V$ is k-dim space and $Stab(V_0)$ is parabolic subspace of $GL(V)$
  - note that $GL(V)$ acts transitively on k-dim subspaces of V.
- $\mathbb{P}^n = Gr(1,\mathbb{C}^{n+1})$ and $\mathbb{P}(V) = Gr(1,V)$

⇝ we can also calculate the dimension using this:

- $\dim GL(V) = n^2$
- for $Stab(V_0)$, choose basis for V_0 and extend to V. Write matrices w/ them so that $V_0 \leftrightsquigarrow (\underbrace{1 \ddots 1}_{k}\ \underbrace{0}_{n-k})$

and $GL(V) \simeq GL_n(\mathbb{C})$ acts by right mult. Stabilising elements of $GL(V)$ have

$$\begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{pmatrix} \underbrace{\left(\begin{array}{cc} \overbrace{\boxed{GL_k(\mathbb{C})}}^{k} & \overbrace{\boxed{0}}^{n-k} \\ \boxed{\text{anything}} & \end{array}\right)}_{n} \begin{array}{l} \}k \\ \}n-k \end{array}$$

which is diminension $n^2 - k(n-k)$.

Hence $\dim Gr(k,\mathbb{C}^n) = n^2 - (n^2 - k(n-k)) = k(n-k)$

② Schubert cells and Schubert varieties

Given placement of 1's, we know which components are free ...

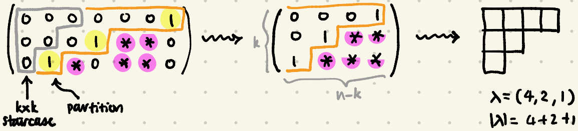
DEF A Schubert cell is

$$\Omega^\circ_{\{i_1,\dots,i_k\}} = \left\{ W \in Gr(k,\mathbb{C}^n) : W \text{ has rref matrix w/ shape } \begin{array}{c} \begin{pmatrix} & & & & 1 \\ & & & \dots & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & & & \uparrow \\ i_1 & i_2 & & \dots & i_k \end{array} \end{array} \right\}$$

eg $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 & -5 & 2 & 0 \end{pmatrix}$ represents space belonging to $\Omega^\circ_{\{2,4,7\}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{pmatrix}$

We have $G(k,n) = \bigcup_{\{i_1,\dots,i_k\} \subseteq [n]} \Omega^\circ_{\{i_1,\dots,i_k\}}$.

We can alternatively index Schubert cells with partitions!



$\lambda = (4,2,1)$

$|\lambda| = 4+2+1 = 7$

- Denote by Ω°_λ, which we can see now has dimension $k(n-k) - |\lambda|$
- More explicit definition: $\Omega^\circ_\lambda := \{ W \in Gr(k,\mathbb{C}^n) : \dim(W \cap \langle e_1,\dots,e_r\rangle) = i \text{ for } n-k+i-\lambda_i \le r \le n-k+i-\lambda_{i+1}, \text{ for all } i \}$

  where $e_1 = (0,\dots,0,1)^T$, $e_2 = (0,\dots,0,1,0)^T$, ... etc. and $\lambda = (\lambda_1, \lambda_2, \dots)$

  accounts for staircase ↑ ↑ accounts for partition eg. (for above example) $n-k=4$, $\lambda = (\underset{\lambda_1}{4}, \underset{\lambda_2}{2}, \underset{\lambda_3}{1})$

$$\begin{array}{r|ccccc} i & 1 & & 2 & & 3 \\ n-k+i-\lambda_i & 1 & \overset{+1}{\nearrow} & 4 & \overset{+1}{\nearrow} & 6 \\ n-k+i-\lambda_{i+1} & 3 & & 5 & & 7 \end{array} \qquad \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

DEF A <u>Schubert variety</u> is , for λ inside $k(n-k)$ square,

$$\Omega_\lambda := \{W \in Gr(k,\mathbb{C}^n) : \dim(W \cap \langle e_1, \dots, e_{n-k+i-\lambda_i}\rangle) \geq i, \text{ for all } i\}$$

Alternatively, $\Omega_\lambda = \overline{\Omega^\circ_\lambda}$ w/ the projective variety structure.

This has dimension $\dim \Omega_\lambda = \dim \Omega^\circ_\lambda = k(n-k) - |\lambda|$

eg. take $\Omega_{\square\square}$ in $Gr(1,\mathbb{C}^6) = \mathbb{P}^5$

- There is one condition defining $\Omega_{\square\square}$:

  $\dim(W \cap \langle e_1, \dots, e_4\rangle) \geq 1$

- So W can be of form

$(\boxed{0}\ \overbrace{0\ 0\ 0\ 1}^{e_1,\dots,e_4})$ $\square\square\square\square\square$

$(\boxed{0}\ 0\ 0\ 0\ 1\ *)$ $\square\square\square\square$

$(\boxed{0}\ 0\ 0\ 1\ *\ *)$ ⟷ $\square\square\square$

$(\boxed{0}\ 0\ 1\ *\ *\ *)$ $\square\square$

~~$(\boxed{0}\ 1\ *\ *\ *\ *)$ $\square$~~

~~$(\boxed{1}\ *\ *\ *\ *\ *)$ $\emptyset$~~

- Hence $\Omega_{\square\square} = \Omega^\circ_{\square\square} \sqcup \Omega^\circ_{\square\square\square} \sqcup \Omega^\circ_{\square\square\square\square} \sqcup \Omega^\circ_{\square\square\square\square\square}$

<u>FACT</u> Schubert varieties are always disjoint unions of Schubert cells

<u>Rk</u> We can replace the definition with any complete flag in place of

$$0 \subset \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \dots \subset \langle e_1, \dots, e_r\rangle \subset \dots \subset \langle e_1, \dots, e_n\rangle = \mathbb{C}^n$$

To get $\Omega^\circ_\lambda(F_\bullet) := \{W \in Gr(k,\mathbb{C}^n) : \dim(W \cap F_r) = i \text{ for } n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1}, \text{ for all } i\}$

and $\Omega_\lambda(F_\bullet) := \{W \in Gr(k,\mathbb{C}^n) : \dim(W \cap F_{n-k+i-\lambda_i}) \geq i, \text{ for all } i\}$

$= \overline{\Omega^\circ_\lambda(F_\bullet)}$.

eg. $\Omega_\square(F_\bullet)$ in $Gr(2,\mathbb{C}^4)$ consists of 2-dim subspaces $V \subseteq \mathbb{C}^4$

such that $\dim(V \cap F_2) \geq 1$.

- ie. "planes in $\mathbb{C}^4$ intersecting plane F_2 in at least a line"
- Going to $\mathbb{P}^3$ instead, this is

  "projective lines intersecting line F_2 at at least a point"

⇝ $|\Omega_\square(F_\bullet^{(1)}) \cap \Omega_\square(F_\bullet^{(2)}) \cap \dots \cap \Omega_\square(F_\bullet^{(r)})| = \#$ lines nontrivially intersecting all lines $\{F_2^{(i)}\}_{i=1}^r$ where $F_\bullet^{(i)}$ generic flags

form open dense subset of flag variety

<u>Application</u>: Littlewood-Richardson rule

X^{even}

- $Gr(k,\mathbb{C}^n)$ is a CW-complex with cells given by Schubert cells
  - attaching maps given by closure relations
  - X^{2i} is given by $\displaystyle\bigsqcup_{\substack{\lambda: \text{remove } i \\ \text{boxes from} \\ \square}} \Omega^\circ_\lambda$
  - eg $Gr(2,\mathbb{C}^4) = \underbrace{\Omega^\circ_{\substack{\square\square\\\square\square}}}_{X^0} \sqcup \underbrace{\Omega^\circ_{\substack{\square\square\\\square}}}_{\substack{X^2 \\ \text{rem } 1\,\square}} \sqcup \underbrace{\Omega^\circ_{\square\square} \sqcup \Omega^\circ_{\substack{\square\\\square}}}_{\substack{X^4 \\ \text{rem } 2\,\square}} \sqcup \underbrace{\Omega^\circ_{\square}}_{\substack{X^6 \\ \text{rem } 3\,\square}} \sqcup \underbrace{\Omega^\circ_{\emptyset}}_{\substack{X^8 \\ \text{rem } 4\,\square}}$

- <u>THM</u> The cohomology ring $H^*(Gr(k,\mathbb{C}^n))$ has $\mathbb{Z}$-basis

$$\sigma_\lambda := [\Omega_\lambda(F_\bullet)] \in H^{2|\lambda|}(Gr(k,\mathbb{C}^n))$$

for λ fitting in $k \times (n-k)$ rectangle.

We have

$$\sigma_\lambda \cdot \sigma_\mu = [\Omega_\lambda(F_\bullet) \cap \Omega_\mu(E_\bullet)] \in H^{2|\lambda|+2|\mu|}(Gr(n,k))$$

where $F_\bullet$ is std. flag, $E_\bullet$ is opp. flag.

We can state the intersection problem with this:

- Let $\lambda^{(1)}, \dots, \lambda^{(r)}$ be partitions, $\sum_i |\lambda^{(i)}| = k(n-k)$, and B be $k \times (n-k)$ partition then
$$|\Omega_{\lambda^{(1)}}(F_\bullet^{(1)}) \cap \dots \cap \Omega_{\lambda^{(r)}}(F_\bullet^{(r)})|$$
is a fin. number $c^B_{\lambda^{(1)},\dots,\lambda^{(r)}}$ of points which is the coeff
$$\sigma_{\lambda^{(1)}} \cdots \sigma_{\lambda^{(r)}} = c^B_{\lambda^{(1)},\dots,\lambda^{(r)}} \sigma_B \in H^{k(n-k)}(Gr(k,\mathbb{C}^n)) = \langle \sigma_B \rangle$$

- Define $\Lambda_{\mathbb{C}}(x_1, x_2, \dots)$ to be space of symmetric functions
$$x_T = x_1^{m_1} x_2^{m_2} \cdots \quad \text{for } T \text{ a SSYT and } m_i = \# \text{ of } i\text{'s in } T$$
$$s_\lambda = \sum_{\substack{T \\ \text{SSYT} \\ \text{shape } \lambda}} x_T \quad \text{(Schur function)}$$

Then $\{s_\lambda\}$ is basis of $\Lambda_{\mathbb{C}}(x_1, x_2, \dots)$, for λ ranging over all partitions

THM $H^*(Gr(k,\mathbb{C}^n)) \simeq \Lambda_{\mathbb{C}}(x_1, x_2, \dots) / \{s_\lambda \mid \lambda \notin k \times (n-k) \text{ grid}\}$
$$\sigma_\lambda \longleftrightarrow s_\lambda$$

$\rightsquigarrow$ 34 223 12 1 11

for every suffix:
$\#_i \geq \#_{i+1}$, $\forall i$

THM (Littlewood-Richardson rule, 1934)

We have $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$

where $c^\nu_{\lambda\mu}$ is # of Littlewood-Richardson tableau w/ shape ν/λ and content μ.

COR

$c^B_{\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)}} = \#$ of chains of Littlewood-Richardson tableau w/ content $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)}$ and total shape $B = k \times (n-k)$ box

eg. If $\lambda^{(i)} = \square = (1)$ for all i, all content are 1, so we only care about filling B in order.

ie, $c^B_{\lambda^{(1)},\dots,\lambda^{(r)}} = \#$ standard Young Tableau shape B
$$= \frac{|B|!}{\prod_{s \in B} \text{hook}(s)} \quad \text{(Hook length formula)}$$

(hook(s) = # boxes in hook of s)

If $k=2$, = # lines nontrivially intersecting all lines $\{F_2^{(i)}\}_{i=1}^r$ for generic flags $\{F_\bullet^{(i)}\}_{i=1}^r$

For example, in $H^*(Gr(2,\mathbb{C}^4))$,
$$\sigma_\square \cdot \sigma_\square \cdot \sigma_\square \cdot \sigma_\square = c^{\boxplus}_{\square,\square,\square,\square} \cdot \sigma_{\boxplus}$$
where $c^{\boxplus}_{\square,\square,\square,\square} = \#$ std Young tableau shape $\boxplus$
$$= \#\left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \right\}$$
$$= 2.$$

So there are 2 lines that non-trivially intersect 4 generic lines in $\mathbb{P}^3$

④ Plücker coordinates and Plücker relations

Our aim is to show $Gr(k,V)$ is a projective subvariety of $\mathbb{P}(\Lambda^k V)$.

Plücker coordinates

- Define a map "Plücker embedding"

$$\begin{aligned} \iota : G(k,V) &\longrightarrow \mathbb{P}(\Lambda^k V) \\ \underset{\langle w_1,\dots,w_k\rangle}{\overset{\shortparallel}{W}} &\longmapsto [w_1\wedge\dots\wedge w_k] \end{aligned}$$

This is well defined because two bases of W are related by change of basis, so the wedge prod. are multiples by det of this change of basis.

PROP ι is injective

Proof It is sufficient to produce left inverse $\varphi : \mathbb{P}(\Lambda^k V) \longrightarrow Gr(k,V)$ of ι.

- Define $\varphi([w]) = \{v\in V : v\wedge w = 0 \in \Lambda^{k+1}V\}$. We want $\varphi\circ\iota(W) = W$ for $W\in Gr(k,V)$.
- If W has basis $\{w_1,\dots,w_k\}$ then $\varphi\circ\iota(W) = \varphi(w_1\wedge\dots\wedge w_k)$ which clearly contains each w_i. So $W \subset \varphi\circ\iota(W)$.
- Next let $v\in\varphi\circ\iota(W)$. Extend basis $\{w_1,\dots,w_k\}$ of W to basis $\{w_1,\dots,w_n\}$ of V. So that $v = \sum_{i=1}^n a_i w_i$, then $0 = v\wedge w_1\wedge\dots\wedge w_k = \sum_{i=1}^n a_i w_i \wedge w_1\wedge\dots\wedge w_k = \sum_{i=k+1}^n a_i\, w_i\wedge w_1\wedge\dots\wedge w_k$

  so $a_i = 0$ for $i \geq k+1$ since the vectors are linearly independent.

  Hence $v = \sum_{i=1}^k a_i w_i \in W$, so $\varphi\circ\iota(W) \subseteq W$. □

(Many of the following lemmas are proved similarly)

If we fix a basis for V to identify $V = \langle v_1,\dots,v_n\rangle \simeq \mathbb{C}^n$ and $\mathbb{P}(\Lambda^k V) \simeq \mathbb{CP}^{\binom{n}{k}-1}$ w/ basis $v_{i_1}\wedge\dots\wedge v_{i_k}$, $1\leq i_1<\dots<i_k\leq n$

then we can explicity write down coordinates of $\iota(W)$ in $\mathbb{P}(\Lambda^k V)$.

That is, if $W = \langle w_1,\dots,w_k\rangle$ then $\iota(W) = [w_1\wedge\dots\wedge w_k]$ where

$$\begin{aligned} w_1\wedge\dots\wedge w_k &= (a_{11}v_1+\dots+a_{n1}v_n)\wedge(a_{12}v_1+\dots+a_{n2}v_n)\wedge\dots\wedge(a_{1n}v_1+\dots+a_{nn}v_n) \\ &= \sum_{0\leq i_1<\dots<i_k\leq n} \underbrace{\sum_{\sigma\in S_k} sgn(\sigma)\, a_{i_1,\sigma(1)}\cdots a_{i_k,\sigma(k)}}_{\det\left(\underbrace{\begin{pmatrix} w_1\\ \vdots \\ w_k\end{pmatrix} \text{ with columns } i_1,\dots,i_k}_{\text{a } k\times k \text{ minor}}\right)} v_{i_1}\wedge v_{i_2}\wedge\dots\wedge v_{i_k} \qquad \text{"Plücker coordinates"} \end{aligned}$$

These are well defined bc. changing basis just scales det of minors by det(row operations) $\neq 0$.

eg. in $Gr(2,\mathbb{C}^4)$ the space w/ matrix

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & -3 & 0 & 3 \end{pmatrix}$$

has Plücker coordinates:

$$\left[\det\begin{pmatrix}0&0\\1&-3\end{pmatrix} : \det\begin{pmatrix}0&1\\1&0\end{pmatrix} : \det\begin{pmatrix}0&2\\1&3\end{pmatrix} : \det\begin{pmatrix}0&1\\-8&0\end{pmatrix} : \det\begin{pmatrix}0&2\\-3&3\end{pmatrix} : \det\begin{pmatrix}1&2\\0&3\end{pmatrix}\right]$$

$$= [0 : -1 : -2 : 3 : 6 : 3]$$

Gr(k,V) as projective variety

We can see $Gr(k,V)$ as projective variety inside $\mathbb{P}(\Lambda^k V)$ by being roots of some equations.

DEF Let V f.d. v.s. and $w\in\Lambda^k V$. Say v divides w if $\exists\, \phi\in\Lambda^{k-1}V$ s.t. $w = v\wedge\phi$

LEM v divides $w \Leftrightarrow v\wedge w = 0$

Proof straight forward

Denote by $D_w = \{v\in V : v\wedge w = 0\}$

DEF Say $w\in\Lambda^k V$ is <u>totally decomposable</u> if $w = v_1\wedge\dots\wedge v_k$ s.t. $\{v_1,\dots,v_k\}\subseteq V$ is lin. indep

Such multivectors will characterise the image of $Gr(k,V)$ inside $\mathbb{P}(\Lambda^k V)$. We write a few characterisations to lead up to this.

PROP $w \in \Lambda^k V$ is totally decomposable $\iff \dim D_w = d$

Proof straightforward

Define map $\varphi : \Lambda^k V \longrightarrow \mathrm{Hom}(V, \Lambda^{k+1} V)$

$$w \longmapsto (v \longmapsto w \wedge v)$$

COR $w \in \Lambda^k V$ totally decomp. $\iff \varphi(w)$ has rank $n-k$

$\iff \ker \varphi(w)$ has dimension k

Proof $\ker(\varphi(w)) = D_w$ which has dim $= d$ ☐

COR $w \in \Lambda^k V$ totally decomp $\iff \varphi(w)$ has rank $\leq n-k$

LEM The rank of $M \in M_{n\times m}(\mathbb{k})$ is largest integer r st. some $r \times r$ minor does not vanish

Proof Fix r as above. let $\mathrm{rk}\, M = p$.

By assumption, there is $r \times r$ minor that doesn't vanish.

The r columns of M that this takes from must be lin. indep so $p \geq r$.

Conversely, take p lin. indep columns of M. Its row rank must also be p, so take p lin. indep rows. This is a non-zero $p\times p$ minor so $p \leq r$. ☐

Almost there:

LEM $[w] \in \mathbb{P}(\Lambda^k V) \cap \iota(Gr(k,V)) \iff w$ is totally decomp

Proof

$(\Leftarrow)$ let $w = v_1 \wedge \dots \wedge v_k$, then $\langle v_1, \dots, v_k \rangle \leq V$ has dimension k.

So $\langle v_1, \dots, v_k \rangle =: U \in Gr(k,V)$ with $\iota(U) = [w]$.

$(\Rightarrow)$ We have $[w] = [w_1 \wedge \dots \wedge w_k] = \iota(W)$ where $W = \langle w_1, \dots, w_k \rangle$ ☐

THM $p(Gr(k,V)) \leq \mathbb{P}(\Lambda^k V)$ is a projective variety

PROOF clearly $\varphi : \Lambda^k V \longrightarrow \mathrm{Hom}(V, \Lambda^{k+1} V)$ is linear.

- For $w \in \Lambda^k V$, think of $\varphi(w)$ as $\binom{n}{k+1} \times n$ matrix w/ entries function of w.
  $\uparrow$ indeterminant
- Then linearity $\varphi(\lambda w) = \lambda \varphi(w)$ shows these functions are homogeneous deg $= 1$.
- We know $[w] \in \iota(Gr(k,V)) \iff \varphi(w)$ has rank $\leq n-k$

$\iff$ all $n-k+1$ minors vanish

ie. $\iota(Gr(k,V))$ is defined by vanishing of $n-k+1$ minors of matrix $\varphi(w)$

$\uparrow$ homogeneous polys

This is not a nice set of polynomials though (they don't generate the ideal

$I(Gr(k,V))$

of poly vanishing on $Gr(k,V)$)

Plücker relations

(shafarevich)

DEF (convolution)

Define for all r,s $\quad - \lrcorner - : \Lambda^r V^* \times \Lambda^s V \longrightarrow \Lambda^{s-r} V$

recursively by

$u^* \in V^* \quad v \in \mathbb{C}, \qquad u^* \lrcorner v = 0$

$u^* \in V^*, \quad v \in V, \qquad u^* \lrcorner v = u^*(v)$

the "$\wedge$" is dropped if one side is just a scalar

$u^* \in V^*, \quad v \in \Lambda^r V, w \in \Lambda^s V \qquad u^* \lrcorner (v \wedge w) = (u^* \lrcorner v) \wedge w + (-1)^r (v \wedge (u^* \lrcorner w))$ (graded leibniz rule)

$u^* = u_1^* \wedge \dots \wedge u_r^* \in \Lambda^r V^*, \quad v \in \Lambda^s V, \qquad u^* \lrcorner v = u_1^* \lrcorner (u_2^* \lrcorner \dots (u_r^* \lrcorner v) \dots))$

PROP $x \in \Lambda^k V$ totally decomp $\iff$ $(y \lrcorner x) \wedge x = 0$ for all $y \in \Lambda^{k-1} V^*$

Proof

($\Rightarrow$) straight forward: for $x = f_1 \wedge \cdots \wedge f_k$

$y \lrcorner x$ is sum of terms in $\Lambda^{k-(k-1)} V \cong V$ where each term is a multiple of some f_i, $1 \le i \le k$.

then it is clear that $(y \lrcorner x) \wedge x = 0$

($\Leftarrow$)
- We prove the contrapositive. Suppose x is totally decomposable.
- Then write x in terms of some fixed basis $f_1, \dots, f_n$ of V.
- Take any non-zero term $f_{i_1} \wedge \cdots \wedge f_{i_k}$ and consider $y = f_{i_1}^* \wedge \cdots \wedge \widehat{f_j^*} \wedge \cdots \wedge f_{i_k}^*$, $j \in \{i_1, \dots, i_k\}$

①: where f_j is not a factor of some other term (guaranteed bc. x has at least 2 different terms)

②: WLOG, there are no other terms w/ $f_{i_1}, \dots, \widehat{f_j}, \dots, f_{i_k}$ bc we can add them all up and replace f_j in basis w/ this sum.

- Then $(y \lrcorner x) = c f_j$ where $c \neq 0$ is coeff of the term $f_{i_1} \wedge \cdots \wedge f_{i_k}$ in x, and $y \lrcorner$(other terms) are zero by ②

Hence $(y \lrcorner x) \wedge x = c f_j \wedge (\underbrace{\text{terms of } x \text{ w/o } f_j}_{\text{not zero by ①}}) \neq 0$

$\Rightarrow \exists y \in \Lambda^{k-1} V$ s.t $(y \lrcorner x) \wedge x \neq 0$

Picking a basis $e_1, \dots, e_n$ for V, the right side is equivalent to checking all $y = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$ in dual basis $\{e_1^*, \dots, e_n^*\}$ of V

The formula can then be rewritten as

PROP $x = \sum_{0 \le i_1 < \cdots < i_k \le n} a_{i_1, \dots, i_k} e_{i_1, \dots, i_k} \in \Lambda^k V$ totally decomp $\iff$ $\sum_{l=1}^{k+1} (-1)^l a_{i_1, \dots, i_{k-1}, j_l} a_{j_1, \dots, \hat{j_l}, \dots, j_{k+1}} = 0$ for all $0 \le i_1 < \cdots < i_{k-1} \le n$, $0 \le j_1 < \cdots < j_{k+1} \le n$

- These generate the ideal $I(Gr(k, V))$
  - note the polynomials are clearly homogeneous of degree 2.

eg. Let $V = \mathbb{C}^4$ and $k = 2$

Write $a_{ij} = -a_{ji}$, $1 \le i < j \le 4$ for the coords of $\Lambda^2 V$

Then Plücker relations are

i_1	j_1	j_2	j_3	$-\sum_{l=1}^{3} (-1)^l a_{i_1 j_l} a_{\dots \hat{j_l} \dots}$
1	1	2	3	$\cancel{a_{11}}a_{23} - \cancel{a_{12}}\cancel{a_{13}} + \cancel{a_{13}}\cancel{a_{12}}$
1	1	2	4	(similar happens when $j_l = i_1$ for some l)
1	1	3	4	
	⋮			
1	2	3	4	$a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23}$
2	1	3	4	$a_{21} a_{34} - a_{23} a_{14} + a_{24} a_{13}$
3	1	2	4	$a_{31} a_{24} - a_{32} a_{14} + a_{34} a_{12}$

There is only one: $a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} = 0$

In general there are many more, and may not be alg. independent