A Diagram Calculus for a few Type-A Lusztig–Vogan Categories

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Outline

1. Motivation: KL conjecture
2. Diagrammatic LV categories for $\mathrm{SU}(n, 1)$
3. Comparison with what came before

Context and Motivation

Complex (simply-connected) Lie groups

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BGG
Category $\mathcal{O}$

Vermas, Simples

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$\{\delta_s\}, \{b_s\}$

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$\mathcal{D}\mathbb{BS}\mathrm{Bim}(\mathfrak{sl}_3)$ is an $\mathbb{R}$-linear monoidal category with

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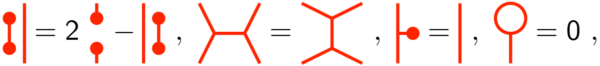
- Local relations up to isotopy



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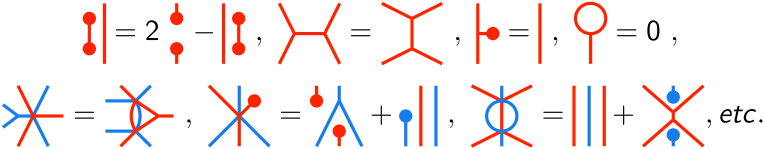

Theorem (Elias–Khovanov 2009)

$\mathcal{D}\mathbb{BS}\mathrm{Bim}(\mathfrak{sl}_3) \simeq \mathbb{BS}\mathrm{Bim}(\mathfrak{sl}_3)$ *as $\mathbb{R}$-linear monoidal categories.*

Diagrammatic Lusztig–Vogan category for $\mathsf{SU}(2,1)$

$$\mathsf{SU}(2,1) = \left\{ M \in \mathsf{SL}_3(\mathbb{C}) : M^* \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}$$

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$\mathcal{D}\tilde{\mathcal{N}}^0(\mathsf{SU}(2,1))$ is an $\mathbb{R}$-linear (right) module category over $\mathcal{D}\mathbb{BS}\mathrm{Bim}(\mathfrak{sl}_3)$

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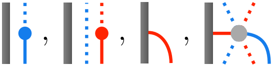
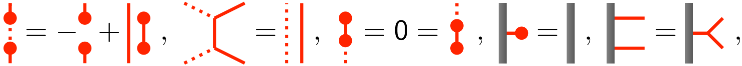
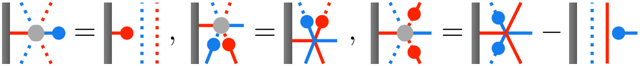


↶

- Local relations up to (limited) isotopy





, *etc.*

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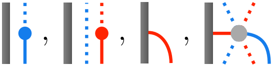

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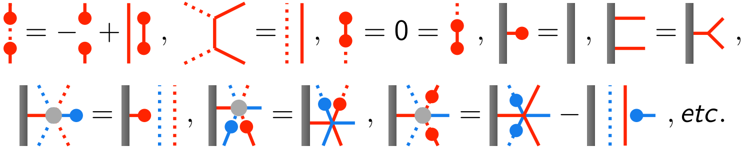
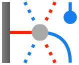


Theorem (Z.)

$\mathcal{D}\tilde{\mathcal{N}}^0(\mathrm{SU}(2,1)) \simeq \tilde{\mathcal{N}}^0(\mathrm{SU}(2,1))$ *as $\mathbb{R}$-linear module categories.*

Diagrammatics in action

E.g.  is an idempotent

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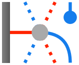
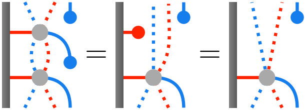
E.g.  is an idempotent



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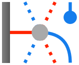
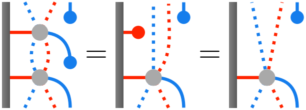
E.g.  is an idempotent



This tells us of the splitting

$$\circ\circ\bullet = \circ\circ(-1) \oplus \circ\circ(1).$$

Diagrammatic Lusztig–Vogan category for $\mathrm{SU}(n,1)$

How do we generalise this to $\mathrm{SU}(n,1)$?

E.g. $\mathcal{D}\tilde{\mathcal{N}}(\mathrm{SU}(3,1))$ with input

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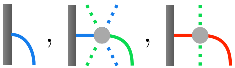

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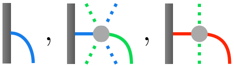
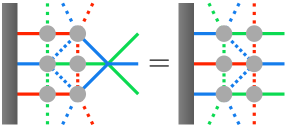


- *New Relation:* with 3-colors



Comparison with what came before

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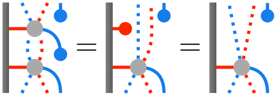
- $\mathcal{D}\tilde{\mathcal{N}}^0(\mathsf{SU}(n,1))$ has a mix of standard bimodule (dotted line) and singular Soergel bimodule (solid line and wall) diagrams.
  - The wall is governed by the parabolic subgroup $W_K \leq W$.
  - The subcategory generated by $\mathbb{1}$ (i.e. solid strands) is the corresponding diagrammatic spherical category.
  - New behaviour occurs where wall and dotted meet, and where solid and dotted meet.



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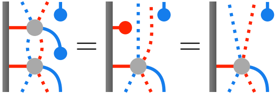

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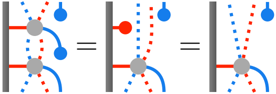

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- For more exotic real groups, $W_K \leq W$ may not even be a reflection subgroup!

Thanks for your attention