(27/02/2026)

Classification of finite dimensional reps of $sl_2(\mathbb{C})$

- We follow lecture 11 of Fulton Harris

Outline

0. Lie algebras and $sl_2(\mathbb{C})$
1. irreducible representations
2. tensors, symmetric powers, exterior powers

0 Lie algebras and $sl_2(\mathbb{C})$

DEF A vector space L over field k, with a "multiplication" $[-,-]: L \times L \to L$ (called <u>bracket</u> or <u>commutator</u>) is a <u>Lie algebra</u> if the following are satisfied

1. $[-,-]$ is bilinear
2. $[x,x]=0$, $\forall x \in L$
3. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$, $\forall x,y,z \in L$ (Jacobi identity)

PROP $[x,y]=-[y,x]$

Proof $0=[x+y,x+y]=[x,x]+[x,y]+[y,x]+[y,y]$
$= [x,y]+[y,x]$ $\square$

Example $sl_2(\mathbb{C}) = \{M \in M_2(\mathbb{C}) : tr(M)=0\}$

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl_2(\mathbb{C})$, $tr(\ )=a+d=0$

  $\rightsquigarrow sl_2(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a,b,c \in \mathbb{C}\right\}$
- $[X,Y]=XY-YX$

1 Irreducible representations

- $sl_2(\mathbb{C})$ is a "semisimple" Lie algebra
  (no non-zero solvable ideals (ie. an algebraic property))

  THM (Weyl) Over an alg. closed field of char $=0$, the f.d. reps of semisimple Lie algebras are completely reducible ($\Leftrightarrow$ all reps decomp. to dir. sum of irreps)

This theorem saves us lots of trouble. W/o it, it is still doable but we need more techniques to eliminate the gen. eigenspaces.

DEF A representation of a Lie algebra $\mathfrak{g}$ is a Lie alg. homomorphism $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ for some vector space V.
Here, picking a basis, $\mathfrak{gl}(V) = \mathfrak{gl}_n(k) = M_n(k)$

- relationship to $SL(2,\mathbb{C})$

  THM Let G be simply conn. Lie group, and $\phi: Lie(G) \to Lie(H)$ is Lie alg. hom. then this lifts uniquely to a Lie group hom. $\tilde{\phi}: G \to H$ s.t. $\phi = d\tilde{\phi}$.

  - In particular $Lie(SL(2,\mathbb{C})) = sl_2(\mathbb{C})$ and $SL(2,\mathbb{C})$ is simply connected.
    So all reps $\rho: sl_2(\mathbb{C}) \to \mathfrak{gl}(V)$ lift uniquely to Lie group rep $\tilde{\rho}: SL(2,\mathbb{C}) \to GL(V)$
  - Of course all Lie group reps can be differentiated to give Lie alg rep, so they are one-to-one.
  - The Lie algebra is <u>easier</u> to study in general bc. we can don't need to worry about differentiability

  RMK This doesn't hold when G is not simply connected.
  eg. $SO(3)$ is simply conn but $SU(2)$ is not
  we have $so(3) \simeq su(2)$ but there is no iso. $SU(2) \to SO(3)$ since $SU(2)$ is a double cover

Actually onto reps...

- $sl_2(\mathbb{C})$ is the easiest case, but it is the backbone for reps of other s.s. Lie algebras so in some sense if you can do $sl_2(\mathbb{C})$, you can do any complex s.s. Lie algebra
- Idea: we find an eigen decomposition of a representation and analyse action on eigenspaces

First some linear algebra

THM (Jordan decomp)

Let V be a f.d. vector space. Any matrix $M: V \to V$ can be written as
$$M = M_s + M_n$$
where M_s is diagonalisable and M_n nilpotent.

A practical corollary of complete reducibility

THM (FH 9.20) Let $\mathfrak{g}$ be a s.s. Lie algebra. For any $X \in \mathfrak{g}$, there exists $X_s, X_n \in \mathfrak{g}$ s.t. for any f.d. rep $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$
$$\rho(X)_s = \rho(X_s) \quad \text{and} \quad \rho(X)_n = \rho(X_n)$$

Proof skipped

- For $sl_2(\mathbb{C})$, there is a natural basis
$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
s.t. $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$

- In the theorem, $H = H_s + H_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the only way to do it to work with the natural rep

If $\rho: sl_2(\mathbb{C}) \to \mathfrak{gl}(V)$ is a rep. then $\rho(H) = \rho(H)_s + \rho(H)_n$

where $\rho(H)_s = \rho(H_s) = \rho\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is diagonalisable

and $\rho(H)_n = \rho(H_n) = \rho(0) = 0$ is nilpotent.

In particular $\rho(H)$ is diagonalisable.

Let $\rho: sl_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be an arbitrary f.d. irred. rep.

We can decompose V into eigenspaces of H
$$V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha \quad \text{where} \quad V_\alpha = \{v \in V : Hv = \alpha v\}$$

We want to use the irreducibility assumption on V, so lets look at action of E, F on these spaces.

- let $v \in V_\alpha$, then
$$\begin{aligned} HEv &= 2Ev + EHv \\ &= (\alpha+2)Ev \quad \Rightarrow Ev \in V_{\alpha+2} \\ HFv &= -2Fv + FHv \\ &= (\alpha-2)Fv \quad \Rightarrow Fv \in V_{\alpha-2} \end{aligned}$$

- So if V_α is an eigenspace for some $\alpha \in \mathbb{C}$, then

$$\bigoplus_{n \in \mathbb{Z}} V_{\alpha+2n}$$
$$\begin{tikzcd} \cdots & V_{\alpha-4} \arrow[l, "F"] \arrow[r, bend left, "E"] \arrow[loop below, "H"] & V_{\alpha-2} \arrow[l, "F"] \arrow[r, bend left, "E"] \arrow[loop below, "H"] & V_\alpha \arrow[l, "F"] \arrow[r, bend left, "E"] \arrow[loop below, "H"] & V_{\alpha+2} \arrow[l, "F"] \arrow[r, bend left, "E"] \arrow[loop below, "H"] & V_{\alpha+4} \arrow[l, "F"] \arrow[r, bend left, "E"] \arrow[loop below, "H"] & \cdots \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & V_\beta \arrow[l, bend left, "F"] \arrow[loop below, "H"] \end{tikzcd}$$

is a subrepresentation of V

- But V is irreducible, so this must equal V
- Also V is finite dimensional, so there must be some β s.t. $V_\beta \neq 0$ and $V_{\beta+2n} = 0$ for all $n \geq 1$

Let $v \in V_\beta$ be non-zero. Consider the set $\{v, Fv, F^2v, \dots\}$

CLAIM $\{v, Fv, F^2v, \dots\}$ spans V

Proof Write W be the subspace gen. by these vectors.
Since V is irreducible, we just need to show W is actually a representation.
Let $w \in W$.
- clearly F preserves W
- also $HF^m v = (\beta - 2m) F^m v$ so H preserves W
- for E, lets do the first few cases

$$Ev = 0$$
$$EFv = Hv + FEv = \beta v$$
$$EF^2 v = HFv + FEFv = (\beta - 2)Fv + F\beta v = (\beta - 2 + \beta) Fv$$

so by induction,

$$\begin{aligned} EF^m v &= HF^{m-1}v + FEF^{m-1}v \\ &= (\beta - 2(m-1))F^{m-1}v + (\beta + \beta - 2 + \dots + \beta + 2(m-2))F^{m-1}v \\ &= (\beta + (\beta - 2) + \dots + (\beta - 2(m-1)))F^{m-1}v \\ &= m(\beta - m + 1)F^{m-1}v \end{aligned}$$

so E preserves W. □

COROLLARY $\dim(V_{\beta - 2n})$ is either 0 or 1
Proof The set $\{v, F\alpha, \dots\}$ is a basis for V and has one vector from each non-zero V_α. □

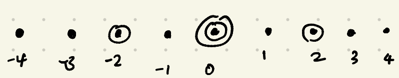
Now we also have (by finiteness) the largest m s.t.

$$V_{\beta - 2(m-1)} \neq 0 \text{ and } V_{\beta - 2m} = 0$$

Then $0 = E0 = EF^m v = m(\beta - m + 1)F^{m-1}v$
so $\beta - m + 1 = 0$ (since $F^{m-1}v \neq 0$ and m cannot be 0)
so $\beta = m - 1$
Particularly β is a non-negative integer and the non-zero V_α's are symmetric about 0.
Hence V looks like

$$\begin{tikzcd} V_{-n} \arrow[r, bend left, "E"] \arrow[loop below, "H"] & V_{-n+2} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop below, "H"] & \cdots & V_{n-2} \arrow[r, bend left, "E"] \arrow[loop below, "H"] & V_n \arrow[l, bend left, "F"] \end{tikzcd}$$

NOTICE
- Each irreducible f.d. rep is determined by the number $n \in \mathbb{Z}_{\geq 0}$
  - since we know how E, F, H acts on all the eigenspaces
  - In fact: there is an irreducible associated to each $n \in \mathbb{Z}_{\geq 0}$ (require checking E, F, H satisfy commutator relations for the action we described)
  - the number n is called the "highest weight" and $V^{(n)} := V$ is called the "irreducible of highest weight n"
- $\dim(V^{(n)}) = n + 1$
- All irred. f.d. reps of $\mathfrak{sl}_2(\mathbb{C})$ have eigenspaces of the same parity (even/odd)
  If a rep has eigenvalues of H w/ multiplicity 1 then it must be irreducible
  $\Rightarrow$ The number of irreducible factors is the sum of multiplicities of 0 and 1 as eigenvals of H.
  eg.



is a rep $V^{(0)} \oplus V^{(2)} \oplus V^{(4)} \oplus V^{(3)}$

We have proved that all irreps are of this form. For the converse, we give an explicit constr. for the irrep of highest weight $n \in \mathbb{Z}_{\geq 0}$

Some background: tensor products and friends

Let V, W be a rep of $\mathfrak{sl}_2(\mathbb{C})$

DEF The tensor $V \otimes W$ is also a rep of $\mathfrak{sl}_2(\mathbb{C})$ by

$$g(v \otimes w) = gv \otimes w + v \otimes gw$$

This is well defined because the action is linear and respects $[-,-]$:

$$\begin{aligned}
[g,h]\cdot(v\otimes w) &= [g,h]v \otimes w + v \otimes [g,h]w \\
&= (gh(v)\otimes w + hg(v)\otimes w) + (v \otimes gh(w) + v \otimes hg(v)) \\
&= gh\cdot(v\otimes w) + hg\cdot(v \otimes w)
\end{aligned}$$

The symmetric power $\mathrm{Sym}^n(V)$ is $V^{\otimes n}$ quotient by $\cdots \otimes v \otimes w \otimes \cdots = \cdots \otimes w \otimes v \otimes \cdots$

The exterior power $\Lambda^n(V)$ is $V^{\otimes n}$ quotient by $\cdots \otimes v \otimes w \otimes \cdots = - \cdots \otimes w \otimes v \otimes \cdots$

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realisation as symmetric powers

- The trivial representation is $V^{(0)} = \mathbb{C}$, where E, F, H all act by 0.
- The standard rep $\mathbb{C}^2$, gen by standard basis $x = \binom{1}{0}$, $y = \binom{1}{0}$, has action by matrix multiplication

  preserves the bracket bc it holds in $\mathfrak{sl}_2(\mathbb{C})$

$$Hx = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x$$

$$Hy = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -y$$

$$\begin{tikzcd} \underset{-1}{\overset{y}{\bullet}} \arrow[r, bend left, "E"] & \underset{1}{\overset{x}{\bullet}} \arrow[l, bend left, "F"] \end{tikzcd}$$

So $\mathbb{C}^2 = \underset{y}{V_{-1}} \oplus \underset{x}{V_1} \simeq V^{(1)}$

- Consider $\mathrm{Sym}^2(\mathbb{C}^2) = \mathbb{C}\langle x^2, xy, y^2 \rangle$.

  product commutes

$$\begin{aligned}
H(x^2) &= x(Hx) + (Hx)x = 2x^2 \\
H(xy) &= x(Hy) + (Hx)y = 0 \\
H(y^2) &= y(Hy) + (Hy)y = -2y^2
\end{aligned}$$

hence $\mathrm{Sym}^2(\mathbb{C}^2) \simeq V^{(2)}$

$$\begin{tikzcd} \underset{-2}{\overset{y^2}{\bullet}} \arrow[r, bend left] & \underset{0}{\overset{xy}{\bullet}} \arrow[l, bend left] \arrow[r, bend left] & \underset{2}{\overset{x^2}{\bullet}} \arrow[l, bend left] \end{tikzcd}$$

- More generally, $\mathrm{Sym}^n(\mathbb{C}^2)$ has

  for $i = 0, \ldots, n$

$$\begin{aligned}
H(x^{n-i}y^i) &= (n-i)(Hx)x^{n-i-1}y^i + i(Hy)x^{n-i}y^i \\
&= (n-2i)x^{n-i}y^i
\end{aligned}$$

So $\mathrm{Sym}^n(\mathbb{C}^2) = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n = V^{(n)}$

$$\begin{tikzcd} \underset{-n}{\overset{y^n}{\bullet}} \arrow[r, bend left] & \underset{-n+2}{\overset{xy^{n-1}}{\bullet}} \arrow[l, bend left] \arrow[r, bend left] & \underset{-n+4}{\overset{x^2y^{n-2}}{\bullet}} \arrow[l, bend left] \arrow[r, bend left] & \cdots & \underset{n-2}{\overset{x^{n-1}y}{\bullet}} \arrow[r, bend left] & \underset{n}{\overset{x^n}{\bullet}} \arrow[l, bend left] \end{tikzcd}$$

- So every irreducible rep is realised as $V^{(n)} = \mathrm{Sym}^n(\mathbb{C}^2)$

③ *Tensor products*

- A basic question: how does $V^{(n)} \otimes V^{(m)}$ decompose into irreducible reps?
- Let's try an example $V^{(2)} \otimes V^{(3)}$

From above,

$$V^{(2)} \otimes V^{(3)} = \mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathrm{Sym}^3(\mathbb{C}^2)$$

e.v.'s: $(-2, 0, 2)$ & $(-3, -1, 1, 3)$

So the eigenvalues of the induced basis in $V^{(2)} \otimes V^{(3)}$ are

$-5, -3, -1, 1,$
$-3, -1, 1, 3,$
$-1, 1, 3, 5$

<u>Eigenvalues in $\otimes$</u>
let $v \in V$, $w \in W$ w/ $Hv = \alpha v$, $Hw = \beta w$
Then $v \otimes w \in V \otimes W$ w/

$$H(v \otimes w) = Hv \otimes w + v \otimes Hw = (\alpha + \beta) v \otimes w$$

which is $V^{(5)} \oplus V^{(3)} \oplus V^{(1)}$

<u>Note</u> the projection $V^{(2)} \otimes V^{(3)} \longrightarrow V^{(5)}$ is $p \otimes q \longmapsto pq$

THM if $a \geq b$

$$V^{(a)} \otimes V^{(b)} = V^{(a+b)} \oplus V^{(a+b-2)} \oplus \cdots \oplus V^{(a-b)}$$

<u>Proof</u>

<u>Claim</u> If V is a rep and write $\sum_\alpha m_\alpha x^\alpha$ where $m_\alpha = \dim V_\alpha$ then $(x - x^{-1}) \sum_\alpha m_\alpha x^\alpha = \sum d_\beta (x^{\beta+1} - x^{-\beta-1})$ where d_β is multiplicity of $V^{(\beta)}$ in V

<u>Proof of claim</u> it is enough to check for V irreducible.

Then $(x - x^{-1})(x^{-n} + x^{-n+2} + \cdots + x^{n-2} + x^n)$

$$= x^{-n+1} + x^{-n+3} + \cdots + x^{n-1} + x^{n+1} - x^{-n-1} - x^{-n+1} - x^{-n+3} - \cdots - x^{n-1}$$

$$= x^{n+1} - x^{-n-1}$$

Claim $\square$

With this,

$$(x - x^{-1})(x^{-a} + x^{-a+2} + \cdots + x^a)(x^{-b} + x^{-b+2} + \cdots + x^b)$$
$$= (x^{a+1} - x^{-a-1})(x^{-b} + x^{-b+2} + \cdots + x^b)$$
$$= (x^{a+b+1} + x^{a+b-1} + \cdots + x^{a+b+1-2b} - x^{-a-b-1} - x^{-a-b+1} - \cdots - x^{-a-b-1+2b})$$

No cancellation bc. $a+b+1, \cdots, a+b+1-2b$ are all positive and the others all negative.
The irreducible factors are $V^{(a+b)}, V^{(a+b-2)}, \cdots, V^{(a-b)}$. $\square$

[Extras]

Symmetric power and exterior power

Let $W = \mathrm{Sym}^2(\mathbb{C}^2) = V^{(2)}$. Q How can we understand $\mathrm{Sym}^n(W)$?

- Easy example: $\mathrm{Sym}^2(W)$
  - W has eigenvalues $-2, 0, 2$
  - $\mathrm{Sym}^2 W$ has eigenvalues $-4, -2, 0, 0, 2, 4$ $\quad (-2 \;\; 0 \;\; 2)$ unordered pairs

•	•	⊙	•	•	$= V^{(4)} \oplus V^{(0)}$
-4	-2	0	2	4	

- Slightly harder example: $\mathrm{Sym}^3(W)$
  - has eigenvalues

$(-2 \;\; 0 \;\; 2)^3$ —

-2	-2	-2	-6	
-2	-2	0	-4	
-2	-2	2	-2	
-2	0	0	-2	
-2	0	2	0	
-2	2	2	2	$V^{(2)}$
0	0	0	0	
0	0	2	2	
0	2	2	4	
2	2	2	6	$V^{(6)}$

$V^{(6)} \oplus V^{(2)}$

THM
$$\mathrm{Sym}^n(\mathrm{Sym}^2(\mathbb{C}^2)) = \bigoplus_{i=0}^{\lfloor n/2 \rfloor} \mathrm{Sym}^{2n-4i}(\mathbb{C}^2)$$

Proof We do another character argument

$$\mathrm{Sym}^n(\mathrm{Sym}^2(\mathbb{C}^2)) \overset{\chi}{\rightsquigarrow} \sum_{i_1 \le i_2 \le \cdots \le i_n} x^{\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_n}}$$

(symmetric polynomials)
where $\lambda_0 = -2$, $\lambda_1 = 0$, $\lambda_2 = 2$
are the eigenvalues of $\mathrm{Sym}^2(\mathbb{C}^2)$

$$= \sum_{\substack{a+c \le n \\ a,c \ge 0}} x^{-2a+2c}$$

$$= \sum_{c=0}^{n} \sum_{a=0}^{n-c} x^{-2a+2c}$$

$$= x^{-2n} + x^{-2n+2} + x^{-2n+4} + x^{-2n+6} + \dots + x^0 \qquad c=0$$
$$+ x^{-2n+4} + x^{-2n+6} + \dots + x^0 + x^2 \qquad c=1$$
$$\cdots$$
$$+ x^{2n-4} + x^{2n-2} \qquad c=n-1$$
$$+ x^{2n} \qquad c=n$$

$\boxed{4i \le 2n \Rightarrow i \le \frac{n}{2}}$

$$= \sum_{i=0}^{\lfloor n/2 \rfloor} \chi(V^{(2n-4i)})$$

□

- Also $\Lambda^2 W$
  - $\Lambda^2 W$ has eigenvalues $-2, 0, 2$ $\quad (-2 \;\; 0 \;\; 2)$ unordered distinct pairs

  so $\Lambda^2 W \simeq W = V^{(2)}$
- Now $\Lambda^3 W$ has eigenvalues 0, so $\Lambda^3 W = V^{(0)}$ trivial rep

  also $\Lambda^n W = 0$ is the zero rep.

- Note that $W \otimes W = V^{(2)} \otimes V^{(2)} = V^{(4)} \oplus V^{(2)} \oplus V^{(0)} = \mathrm{Sym}^2 W \oplus \Lambda^2 W$

  Q does this happen in general?

$$V \otimes V = \mathrm{Sym}^2 V \oplus \Lambda^2 V$$
$$u \otimes v \longmapsto \left(\tfrac{1}{2}(u \otimes v + v \otimes u), \tfrac{1}{2}(u \otimes v - v \otimes u)\right)$$
$$x + y \longleftarrow (x, y)$$